Convexity and Inequality

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- Lavrentiev and Lyapunov were representatives of the greatest scientific school of the tewentieth century, the school of Luzin. The triumphs and tragedies of Luzin's school is a hologram of the triumphs and tragedies of the Soviet Russia. The fates of Lavrentiev and Lyapunov are the worlf lines to good and light through the turbulent fluxes of blood and evil.

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- Many students of Luzin participated in his pursue and political execution. Lavrentiev and Lyapunov never betrayed their teacher and continued his deeds. The Physics and Mathematics School is a juvenile affiliation of of the great Russian mathematical school of Luzin. Remembering our teachers Lavrentiev and Lyapunov, we bow to them and thank their mutual teacher, Nikolai Nikolaevich Luzin.

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- Linear inequality implies linearity and order. When combined, the two produce an ordered vector space. Each linear inequality in the simplest environment of the sort is some half-space. Simultaneity implies many instances and so leads to the intersections of half-spaces. These yield polyhedra as well as arbitrary convex sets, identifying the theory of linear inequalities with convexity.

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- This talk addresses the origin and the state of the art of the relevant areas with a particular emphasis on the Farkas Lemma [2]. Our aim is to demonstrate how Boolean valued analysis may be applied to simultaneous linear inequalities with operators.

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- Convexity traces back to the idea of a solid figure in plane geometry. Book I of Euclid's *Elements* [3] reads:
- Definition 13. A boundary is that which is an extremity of anything.
- Definition 14. A figure is that which is contained by any boundary or boundaries.

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- Definition 9. Similar solid figures *are those contained by similar planes equal in multitude*.
- Definition 10. Equal and similar solid figures *are those contained by similar planes equal in multitude and magnitude.*

The Three Polymaths

- Convexity and inequality stem from the remote ages [5]–[7]. But as the acclaimed pioneers who propounded these ideas and anticipated their significance for the future, we must rank the three polymaths:
- Joseph-Louis Lagrange (January 25, 1736–April 10, 1813)
- Jean-Baptiste Joseph Fourier (March 21, 1768–May 16, 1830)
- Hermann Minkowski (June 22, 1864–January 12, 1909)

Joseph Lagrange (1736–1813)

- In both research and exposition, he totally reversed the methods of his predecessors. They had proceeded in their exposition from special cases by a species of induction; his eye was always directed to the highest and most general points of view. . . . (Thomas J. McCormack [8])

Joseph Fourier (1768–1830)

- He [Fourier] himself was neglected for his work on inequalities, what he called "Analyse indéterminée." Darboux considered that he gave the subject an exaggerated importance and did not publish the papers on this question in his edition of the scientific works of Fourier. Had they been published, linear programming and convex analysis would be included in the heritage of Fourier. (Jean-Pierre Kahane [9])

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- Our science, which we loved above all else, brought us together; it seemed to us a garden full of flowers:.. In it, we enjoyed looking for hidden pathways and discovered many a newperspective that appealed to our sense of beauty and when one of us showed it to the other and we marvelled over it together, our joy was complete. He was for me a rare gift from heaven. . . . and I must be grateful to have possessed that gift for so long. Now death has suddenly torn him from our midst. However, what death cannot take away is his noble image in our hearts and the knowledge that his spirit in us continue to be active. (David Hilbert [10])

Convexity as Abstraction

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- Stretching a rope taut between two stakes produces a closed straight line segment, the continuum in modern parlance. Rope-stretching raised the problem of measuring the continuum. The continuum hypothesis of set theory is the shadow of the ancient problem of harpedonaptae. Rope-stretching independent of the position of stakes is uniform with respect to direction in space. The mental experiment of uniform rope-stretching yields a compact convex figure.

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- Convexity has found solid grounds in set theory. The Cantor paradise became an official residence of convexity. Abstraction becomes an axiom of set theory. The abstraction axiom enables us to reincarnate a property, in other words, to collect and to comprehend. The union of convexity and abstraction was inevitable. This yields abstract convexity [11]–[13].

Environment for Convexity

Environment for Convexity

- Let $\overline{E}$ be a vector lattice E with the adjoint top $\top := +\infty$ and bottom $\bot := -\infty$. Assume further that H is some subset of E that is by implication a (convex) cone in E, and so the bottom of E lies beyond H. A subset U of H is *convex relative to* H or *H-convex* provided that U is the *H-support set* $U_p^H := \{h \in H : h \leq p\}$ of some element p of $\overline{E}$. Limiting finite subsets of H-convex sets yields some analogs of polyhedra.

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- An element $p \in \overline{E}$ is *H-convex* provided that $p = \sup U_p^H$; i.e., p represents the supremum of the H-support set of p. The proper H-convex elements fill the cone $\mathscr{C}(H, \overline{E})$. The *Minkowski duality* $\varphi : p \mapsto U_p^H$ enables us to study convex elements and sets simultaneously.

Lyapunov's Convexity Theorem

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- The celebrated Lyapunov Convexity Theorem had raised the problem of describing the compact convex sets in finite-dimensional real spaces which serve as the ranges of diffuse measures. These compacta are known in the modern geometrical literature as *zonoids*. Among zonoids we distinguish the Minkowski sums of finitely many straight line segments. These sets, called *zonotopes*, fill a convex cone in the space of compact convex sets, and the cone of zonotopes is dense in the closed cone of all zonoids. The description of the ranges of diffuse vector measures in the Lyapunov Convexity Theorem was firstly found by Chuĭkina practically in the modern terms (see[14]). Soon after that her result was somewhat supplemented and simplified by Glivenko in [15]. The zonotopes of the present epoch were called *parallelohedra* those days.

Zonoids

Zonoids

- The significant further progress in studying the ranges of diffuse vector measures belong to Reshetnyak and Zalgaller who described zonoids as the results of mixing the linear elements of a rectifiable curve in a finite-dimensional space in 1954 (see [16]). In this same paper they suggested a new prove of the Lyapunov Convexity Theorem and demonstrated that zonotopes are precisely those convex polyhedra whose two-dimensional faces have centers of symmetry. Unfortunately, these results remained practically unnoticed in the West. Analogous results were obtained by Bolker only fifteen years later in 1969 (see [17], [18]).

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- In more detail: For optimal transition in minimal time from one state of a system to the other in the conditions of limited resources one can use an extreme “bang-bang” control.

Environment for Inequality

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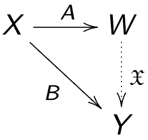
- Assume that X is a real vector space, Y is a *Kantorovich space* also known as a complete vector lattice or a Dedekind complete Riesz space. Let $\mathbb{B} := \mathbb{B}(Y)$ be the *base* of Y, i.e., the complete Boolean algebras of positive projections in Y; and let $m(Y)$ be the universal completion of Y. Denote by $L(X, Y)$ the space of linear operators from X to Y. In case X is furnished with some Y-seminorm on X, by $L^{(m)}(X, Y)$ we mean the *space of dominated operators* from X to Y. As usual, $\{T \leq 0\} := \{x \in X \mid Tx \leq 0\}$; $\ker(T) = T^{-1}(0)$ for $T : X \to Y$. Also, $P \in \mathrm{Sub}(X, Y)$ means that P is *sublinear*, while $P \in \mathrm{PSub}(X, Y)$ means that P is *polyhedral*, i.e., finitely generated. The superscript $^{(m)}$ suggests domination.

Kantorovich's Theorem

[1] Cp. [24, p. 51].

Kantorovich's Theorem

- Find $\mathfrak{X}$ satisfying



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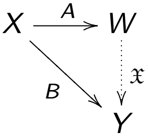

- **(1):** $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.

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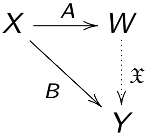

- **(1):** $(\exists \mathfrak{X})\ \mathfrak{X}A = B \leftrightarrow \ker(A) \subset \ker(B)$.
- **(2):** *If W is ordered by W_+ and $A(X) - W_+ = W_+ - A(X) = W$, then*[1]

$$(\exists \mathfrak{X} \geq 0)\ \mathfrak{X}A = B \leftrightarrow \{A \leq 0\} \subset \{B \leq 0\}.$$

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The Farkas Alternative

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- *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume that $A_1, \dots, A_N$ and B belong to $L^{(m)}(X, Y)$. Then one and only one of the following holds:*
  (1) *There are $x \in X$ and $b, b' \in \mathbb{B}$ such that $b' \leq b$ and*

  $$b' Bx > 0, bA_1 x \leq 0, \dots, bA_N x \leq 0.$$

  (2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))_+$ such that $B = \sum_{k=1}^{N} \alpha_k A_k$.*

Boolean Modeling

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Boolean Modeling

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- Takeuti coined the term "Boolean valued analysis" for applications of the models to analysis.[3]

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Scott's Comments

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- Scott forecasted in 1969:[4]
  We must ask whether there is any interest in these nonstandard models aside from the independence proof; that is, do they have any mathematical interest? The answer must be yes, but we cannot yet give a really good argument.

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- In 2009 Scott wrote:[5]

  At the time, I was disappointed that no one took up my suggestion. And then I was very surprised much later to see the work of Takeuti and his associates. I think the point is that people have to be trained in Functional Analysis in order to understand these models. I think this is also obvious from your book and its references. Alas, I had no students or collaborators with this kind of background, and so I was not able to generate any progress.

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  with On the class of all ordinals.
- The truth value $[\![\varphi]\!] \in \mathbb{B}$ is assigned to each formula φ of ZFC relativized to $\mathbb{V}^{(\mathbb{B})}$.

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$$(\exists z \in x{\downarrow})[\![(\exists t \in x)\ \varphi(t)]\!] = [\![\varphi(z)]\!].$$

- The *ascent* functor acts in the opposite direction.

The Reals Within

[6] Cp. [19, p. 349].

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- There is an object $\mathscr{R}$ inside $\mathbb{V}^{(\mathbb{B})}$ modeling $\mathbb{R}$, i.e.,

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- Implement the descent of the structures on $|\mathscr{R}|$ to $\mathscr{R}{\downarrow}$ as follows:

$$x + y = z \leftrightarrow [\![x + y = z]\!] = \mathbb{1};$$
$$xy = z \leftrightarrow [\![xy = z]\!] = \mathbb{1};$$
$$x \leq y \leftrightarrow [\![x \leq y]\!] = \mathbb{1};$$
$$\lambda x = y \leftrightarrow [\![\lambda^{\wedge} x = y]\!] = \mathbb{1}\ (x, y, z \in \mathscr{R}{\downarrow},\ \lambda \in \mathbb{R}).$$

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- **Gordon Theorem.**[6] *$\mathscr{R}{\downarrow}$ with the descended structures is a universally complete vector lattice with base $\mathbb{B}(\mathscr{R}{\downarrow})$ isomorphic to $\mathbb{B}$.*

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- The analogous version of the Farkas Lemma simply fails for two simultaneous inequalities in general.
- The inclusion $\{f = 0\} \subset \{g \le 0\}$ equivalent to the inclusion $\{f = 0\} \subset \{g = 0\}$ does not imply that f and g are proportional in the case of an arbitrary subfield of $\mathbb{R}$. It suffices to look at $\mathbb{R}$ over the rationals $\mathbb{Q}$, take some discontinuous $\mathbb{Q}$-linear functional on $\mathbb{Q}$ and the identity automorphism of $\mathbb{Q}$.

Inhomogeneous Inequalities

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- *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Assume given some dominated operators $A_1, \dots, A_N, B \in L^{(m)}(X, Y)$ and elements $u_1, \dots, u_N, v \in Y$. The following are equivalent:*
  (1) *For all $b \in \mathbb{B}$ the inhomogeneous operator inequality $bBx \le bv$ is a consequence of the consistent simultaneous inhomogeneous operator inequalities $bA_1x \le bu_1, \dots, bA_Nx \le bu_N$, i.e.,*

$$\{bB \le bv\} \supset \{bA_1 \le bu_1\} \cap \dots \cap \{bA_N \le bu_N\}.$$

  (2) *There are positive orthomorphisms $\alpha_1, \dots, \alpha_N \in \operatorname{Orth}(m(Y))$ satisfying*

$$B = \sum_{k=1}^{N} \alpha_k A_k; \quad v \ge \sum_{k=1}^{N} \alpha_k u_k.$$

Sublinear Inequalities

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- *Let X be a Y-seminormed real vector space, with Y a Kantorovich space. Given are some dominated polyhedral sublinear operators $P_1, \dots, P_N \in \mathrm{PSub}^{(m)}(X, Y)$ and a dominated sublinear operator $P \in \mathrm{Sub}^{(m)}(X, Y)$. Assume further that $u_1, \dots, u_N, v \in Y$ make consistent the simultaneous inhomogeneous inequalities $P_1(x) \le u_1, \dots, P_N(x) \le u_N$.*
  The following are equivalent:
  (1) *for all $b \in \mathbb{B}$ the inhomogeneous sublinear operator inequality $bP(x) \ge bv$ is a consequence of the simultaneous inhomogeneous sublinear operator inequalities $bP_1(x) \le bu_1, \dots, bP_N(x) \le bu_N$, i.e.,*

$$\{bP \ge bv\} \supset \{bP_1 \le bu_1\} \cap \dots \cap \{bP_N \le bu_N\};$$

  (2) *there are positive $\alpha_1, \dots, \alpha_N \in \mathrm{Orth}(m(Y))$ satisfying*

$$(\forall x \in X)\ P(x) + \sum_{k=1}^{N} \alpha_k P_k(x) \ge 0, \quad \sum_{k=1}^{N} \alpha_k u_k \le -v.$$

Lagrange's Principle

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- The Slater condition allows us to eliminate polyhedrality as well as considering a unique target space. This is available in a practically unrestricted generality [24].
- About the new trends relevant to the Farkas Lemma see [25]–[29].

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- Freedom presumes liberty and equality. Inequality paves way to freedom.

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