Numerical modeling of generation of waves behind the obstacles in stratified fluid *

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The article deals with the numerical simulation of the stratified incompressible flows. The mathematical model is based on the Boussinesq approximation of the Navier–Stokes equations. The flow field over the sinusoidal hill and behind the moving obstacle is modeled for wide range of Richardson numbers. The resulting set of PDE's is then solved by three different numerical methods.

1. Introduction

It is well known that the influence of the stratification is significant in many processes taking place in ABL (e.g. stratification affects the transport of pollutants, plays significant role in determining the consequences of accidents on environment and human etc.). Stratified flows in environmental applications are characterized by the variation of fluid density in the vertical direction that can lead to appearance of specific phenomena which are not present when density is constant, namely internal and gravity waves, jet–like flow structures, thin interfaces with high density and velocity gradients and anisotropic turbulence. The internal waves are particularly interesting since they effectively transport momentum but not mass. Generation of the internal waves by moving body and behind the hill is studied numerically.

2. Boussinesq approximation

The flow is assumed to be incompressible, yet the density is not constant due to gravity. This type of flow is described by the Navier-Stokes equations for viscous incompressible flow with variable density. These equations are simplified by the Boussinesq approximation. Density and pressure are divided into two parts: a background field (with subscript $_0$) plus a perturbation. The system of equations obtained is partly linearized around the average state ρ_*. The resulting

*This work have been supported by Research Plans MSM 6840770003 and grant TA 01020428.

set of equations can be written in the form

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho u_j)}{\partial x_j} = -u_2 \frac{\partial \varrho_0}{\partial x_2}, \tag{1}$$

$$\frac{\partial u_i}{\partial t} + \frac{\partial(u_j u_i)}{\partial x_j} + \frac{1}{\varrho_*}\frac{\partial p}{\partial x_i} = K \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \delta_{i,2} g, \tag{2}$$

$$\frac{\partial u_j}{\partial x_j} = 0, \tag{3}$$

where $W = [\varrho, u_1, u_2, p]^T$ is the vector of unknowns, $\varrho(x_1, x_2, t)$ denotes the perturbation of the density and u_1, u_2 are the two velocity components, p stands for the pressure perturbation and g for the gravity acceleration. The x_1–axis is orientated in the direction of the motion and the x_2–axis is perpendicular to the density gradient.

For the description of the stratified flows with characteristic velocity U and characteristic length L following parameters have been used:

$$Re = \frac{UL}{\nu}, \qquad Ri = -\frac{g}{\varrho_*} \frac{\frac{\partial \varrho_0}{\partial z}}{U}. \tag{4}$$

3. Numerical schemes

Three different numerical schemes have been used for solution of the mentioned problems. The first scheme is the WENO scheme combined with the projection method, the second scheme is the AUSM MUSCL scheme in the finite volume formulation combined with the artificial compressibility method and the third scheme is the compact finite–difference scheme.

- **WENO scheme combined with projection method**

The first scheme uses the spectral projection to solenoidal field in combination with the WENO scheme for discretization of the inviscid fluxes.

The projection step is suggested in the following form: solve (1)-(3) in the equivalent form

$$\frac{\partial u_i}{\partial t} = P\left(-\frac{\partial(u_j u_i)}{\partial x_j} + K \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \delta_{i,2} g\right), \tag{5}$$

where P is the projection to divergence-free fields. The projection itself is realized in the Fourier space with high order spectral filter. The spatial derivatives and the source term in the momentum equation (2) are achieved by the fifth-order WENO scheme for convective terms and the fourth-order central scheme for the viscous terms. The discretization in time is achieved by the third order TVD Runge–Kutta method.

The equation (1) for density disturbation is solved with the same Runge–Kutta method with WENO approximation for spatial derivatives (without projection).

- **AUSM scheme**

The second scheme is based on the artificial compressibility method in dual time. The continuity equation (3) is rewritten in the form

$$\frac{\partial p}{\partial \tau} + \beta^2 \frac{\partial u_j}{\partial x_j} = 0, \tag{6}$$

where τ is the artificial time. The finite volume AUSM scheme is used for the spatial semidiscretization of the inviscid fluxes. Quantities on the cell faces are computed using the MUSCL reconstruction with the Hemker-Koren limiter. The scheme is stabilized according to (1) by the pressure diffusion. The viscous fluxes are discretized using central approach on a dual mesh (diamond type scheme).

The spatial discretization results in a system of ODE's solved by the second-order BDF formula

$$\frac{3W^{n+1} - 4W^n + W^{n-1}}{2\Delta t} + L^{n+1} = 0. \tag{7}$$

Here, L^{n+1} denotes the numerical approximation of the convective and viscous fluxes described above and the source terms. Arising set of nonlinear equations is then solved by the artificial compressibility method in the dual time τ by the explicit 3-stage second-order Runge-Kutta method.

• **Compact finite–difference scheme**

The third scheme is also based on the artificial compressibility method in dual time. The modified continuity equation (6) is used. The spatial semidiscretization is directly based on the paper (2), where the class of very high order compact finite difference schemes was introduced and analyzed. The main idea used to construct this family of schemes is that instead of approximating the spatial derivatives ϕ' of certain quantity ϕ explicitly from the neighboring values ϕ_i, the (symmetric) linear combination of neighboring derivatives $(\ldots, \phi'_{i-1}, \phi'_i, \phi'_{i+1}, \ldots)$ is approximated by weighted average of central differences. These schemes form a subclass of three-diagonal schemes with five-point computational stencil. The low-pass filter (for the filtered values $\widehat{\phi}_i$) of the following form was used :

$$b\,\widehat{\phi}_{i-1} + \widehat{\phi}_i + b\,\widehat{\phi}_{i+1} = 2\beta_0\phi_i + \beta_1\frac{\phi_{i+1} + \phi_{i-1}}{2h} + \beta_2\frac{\phi_{i+2} + \phi_{i-2}}{4h} + \ldots \tag{8}$$

Resulting system of ODE's has been solved by the three stage second order Strong Stability Preserving Runge-Kutta methods.

All schemes were validated in our previous studies. The schemes have been successfully used for simulation of the flow field around moving bodies in 2D and 3D stratified fluid for wide range of Richardson numbers see (3), (4), (5), (6).

4. Computational setup

The first problem solved in this study is the towing tank problem of dimensions 2.2×0.6 m with moving thin vertical strip 0.025×0.002 m. The strip is located $1m$ from the left wall and at the mid–heights. At the time $t = 0$ the obstacle starts moving to the right (in the positive x_1 direction) with constant velocity $U^{ob} = 0.0026\ m/s$. The flow field is initially at rest with the exponential profile of stratification $\varrho_0 = \varrho_{00} \exp\frac{x_2}{\Lambda}$, $\varrho_{00} = 1008.9\ kg/m^3$, $\Lambda = 38.75$ m, the kinematic viscosity is $\nu = 10^{-6}\ m^2/s$. This computational setup corresponds to the experimental setup of (7) and numerically was studied in (6).

Two sets of boundary conditions have been used in our computations. For the WENO scheme, the periodic boundary conditions for all computed quantities have been used. In the case of the AUSM MUSCL scheme either the same boundary conditions or homogeneous Dirichlet boundary conditions for the velocity and Neumann conditions for the density and pressure disturbances have been prescribed. The computations have been performed on the mesh of 2200×600 cells where one cell corresponds to $1mm$ of the physical domain.

The second computational case is selected as a part of 2D wall-bounded half space with low smooth sine-shaped hill. The hill height is $h = 1m$, while the whole domain has dimensions $90\times$

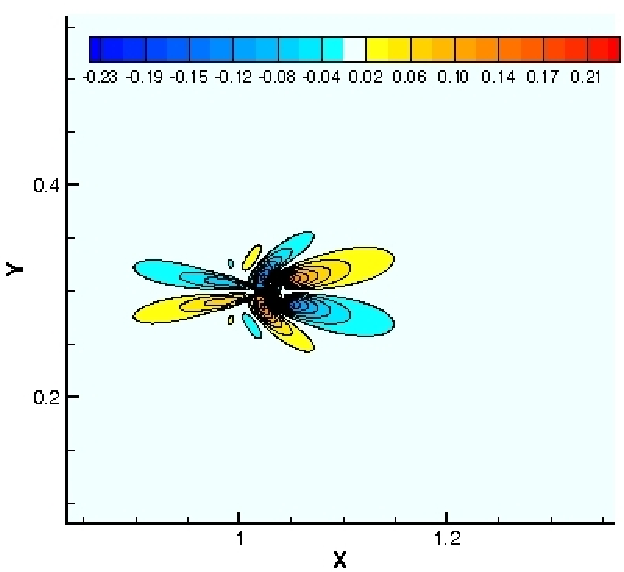
$30\,m$. On the inlet the velocity profile of the form $u_1(x_2) = U_0(x_2/H)^{1/r}$ with $U_0 = 1m/s$ and $r = 40$ was prescribed, density perturbation ρ' is set to zero. Homogeneous Neumann conditions are satisfied on all other boundaries except the wall, where no–slip boundary conditions for velocity components are prescribed.

The background density field is given by $\varrho_0(x_2) = \varrho_w + \gamma x_2$ with $\varrho_w = 1.2\,kg \cdot m^{-3}$ and $\gamma = -0.01\,kg \cdot m^{-4}$, the viscosity $\nu = 0.001$. A range of gravity acceleration g was used to test the behavior of the model and numerical method for different Brunt-Väisälä frequencies. The values used are $g = -2,\ -5,\ -10,\ -20, m \cdot s^{-2}$.
The computations have been performed on structured non–orthogonal grid. The grid consist of 233×117 points refined near the ground and in the vicinity of the hill. The minimal resolution in the x_2 direction is $\Delta x_2 = 0.03m$.

5. Numerical results

Fig. 1 show the process of the wave generation. The flow pattern is typical for transient internal waves past an impulsively started body in stably stratified flow. The thin strip generates an initial perturbation and then gravity waves are formed. The upstream disturbances are pronounced, what is typical for the flow with relatively low Froude number. Behind the obstacle strip with step–like density profile is formed.

In Fig. 2, the comparison of the WENO scheme (left) and the AUSM MUSCL scheme (right) is given. The figure shows isolines of density gradient. This quantity corresponds to the figures obtained by the experimental Schlieren technique. The comparison shows a good qualitative agreement. The more dissipative AUSM MUSCL scheme reproduces one half-wave less in comparison to the WENO scheme.



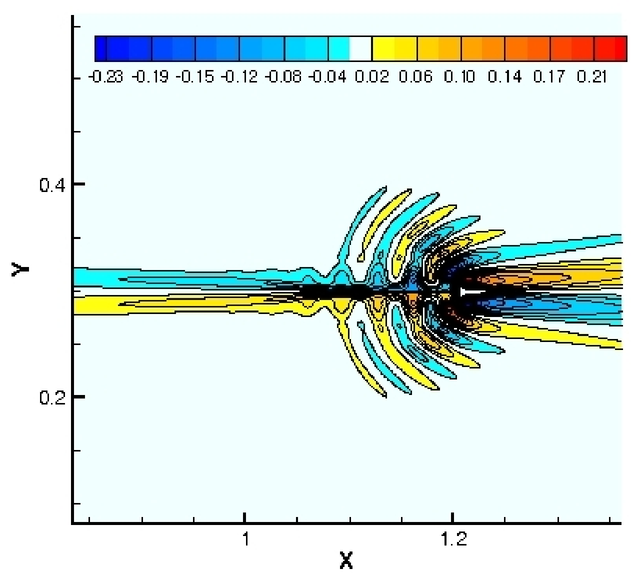


Fig. 1. Evolution of the density perturbance ϱ for two different times $t = 16.5$ and $82.3s$. WENO scheme.

Fig. 3 displays the dependence of the flow on the level of stratification for the hill case. A comparison of the isolines of the u_2–velocity component for four different Richardson numbers ($Ri = 0.2, 0.5, 1, 2$) is presented at the same time. The gravity waves with the wavelength given

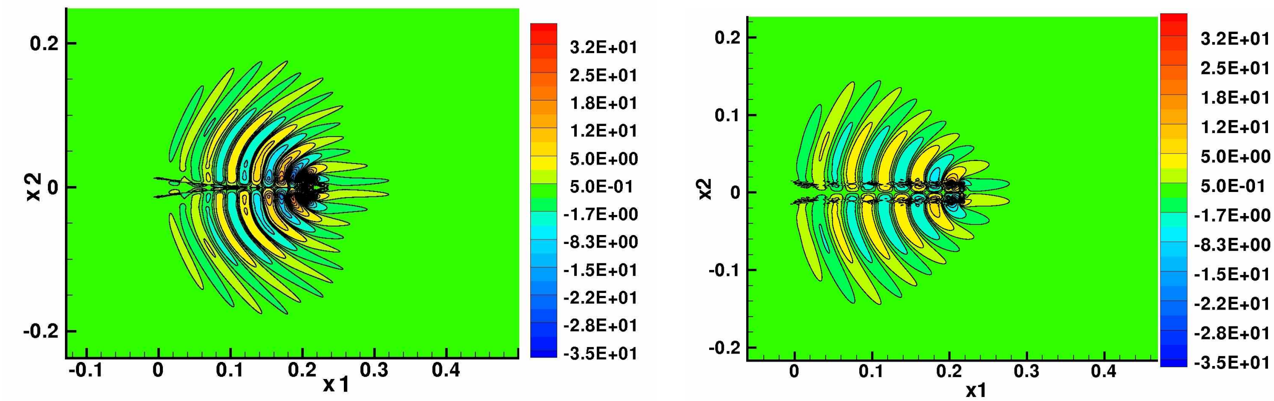


Fig. 2. Isolines of the density gradient $\frac{\partial \varrho}{\partial x_1}$ at the time $t = 82s$. WENO scheme left, AUSM MUSCL right.

by the Brunt–Väisälä frequencies are visible. The presented simulations are affected by some non-physical artifacts related to the implementation of the boundary conditions on the artificial boundaries of the computational domain. Most pronounced is the wave pattern located close to the lower left corner of the domain.

Fig. 4 left demonstrates dependency of the gravity waves on Ri number. The correct resolution of the gravity waves is sensitive to the stabilization term. Relatively small pressure diffusion leads not only to dumping of the waves but also to changes in its frequency as shown in Fig. 4 right.

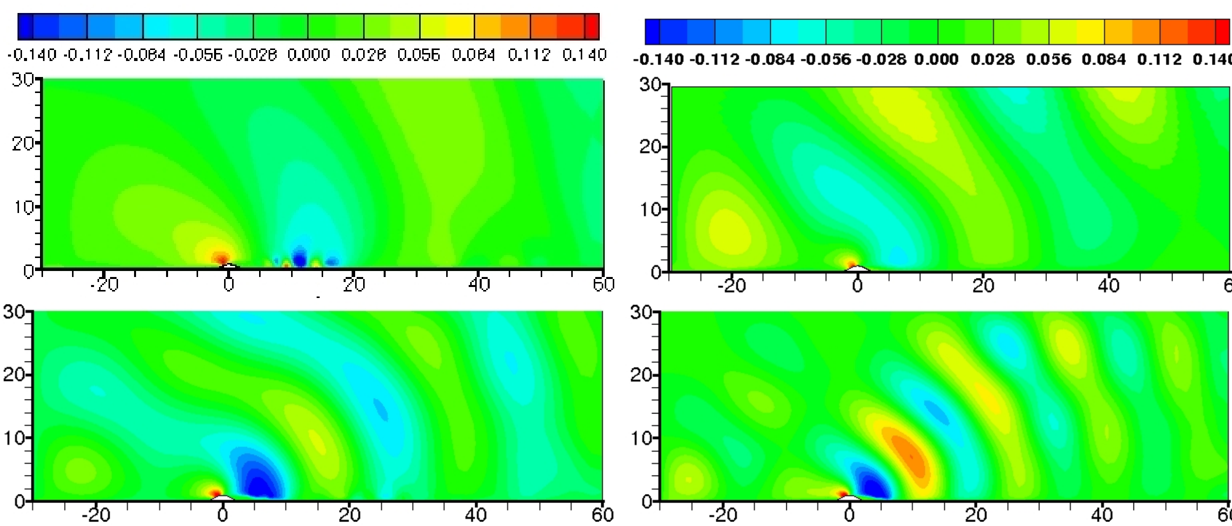


Fig. 3. Isolines of the u_2- velocity component for $Ri = 0.2, 0.5, 1, 2,$ AUSM MUSCL scheme.

6. Conclusion

Three numerical schemes for stratified flows have been developed and they have been successfully used for simulation of the flow around the moving body and over sinusoidal hill. Several numerical results for different Richardson numbers have been obtained. The influence of pressure diffusion term has been demonstrated.

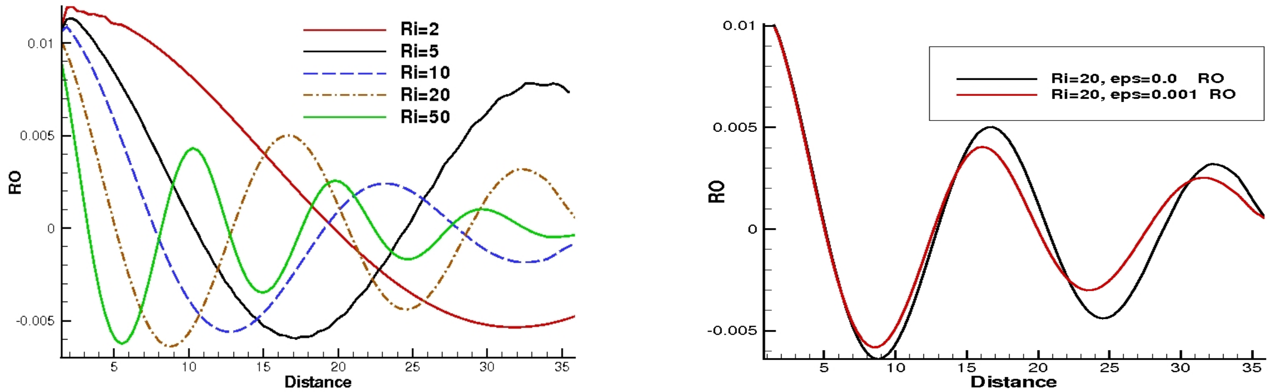


Fig. 4. Dependance of the gravity waves on the *Ri* number (left) and on the pressure diffusion term (right), AUSM MUSCL scheme.

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