Fast Rough Bounds for the Coefficients of the Network Reliability Polynomial

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What is Reliability Polynomial?

When considering a random graph with reliable nodes and unreliable edges that can fail independently, the Reliability Polynomial (RP) is the equation for the selected reliability indicator in the case of equal reliability of the edges. Most explored are ATR polynomials, which means RP of all-terminal reliability (probability of a graph being connected).

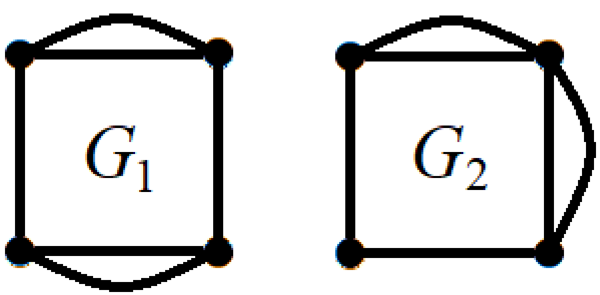
$$R(G,p) = \sum_{H \in \Gamma} P(H)\delta(H),$$

$$\delta(H) = \begin{cases} 1, & \text{if } H \text{ is connected}, \\ 0, & \text{otherwise}. \end{cases}$$

Why we examine RPs?

Reliability polynomials helps in structural optimization of networks and finding bottlenecks in their structure.

They help in proving non-isomorphism of graphs also: **isomorphic graphs have equal reliability polynomials, the converse is not true.**



$$R(G_1)=R(G_2)=p^6+6p^5q+14p^4q^2+12p^3q^3;$$
$$R(G_3)=p^6+6p^5q+12p^4q^2+10p^3q^3.$$

Representation of RP

$$R(G,p)=\sum_{i=0}^{m-n+1} a_i p^{m-i}(1-p)^i$$

Meaning of a_i – the number of connected sugraphs of G, that may be obtained by removing exactly i edges.

We directly obtain:

1. $a_0=1$.
2. $\forall_{i>m\text{-}n+1}\ a_{\mathrm{i}}=0$.

It is obvios also:

3. $a_{\mathrm{m-n+1}}$=<the number of covering trees>.
4. $\forall_{0\leq i<k} = C_m^i$, where k – edge connectivity.

Problem's complicity

That the task of obtaining coefficients of RP for ATR is NP-hard is well-known (Oxley J. and Welsh D. Chromatic, flow and reliability polynomials: The complexity of their coefficients. Comb. Probab. Comput., 11(4):403–426, July 2002.) In general case, for obtaining all a_i we need check 2^m variants of a graph's destructions, where m – number of its edges. Thus, obtaining their exact values is UNREAL(!!!)

Bounds of coefficients

According to their meaning, all a_i are non-negative. If we find for each a_i some b_i и u_i, such, that, $0 \le b_i \le a_i,\ a_i \le u_i \le C_m^i$ then corresponding polynomials $B(G,p)$ and $U(G,p)$ are minorizing and majorising for $R(G,p)$.

Therefor, our task is finding b_i and u_i closest possible to a_i in minimal possible time.

What information we can obtain fast enough?

We assume, that prearrangement for examining graph G includes finding:

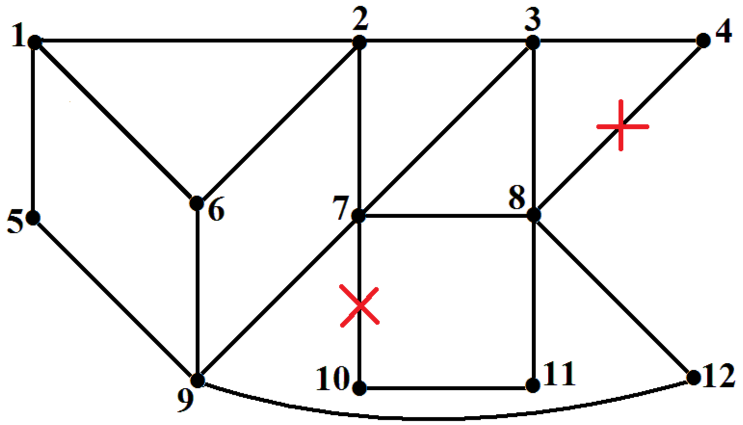
1. All chains that are not bridges, storing their lengths (l_i) and numbers of terminal nodes. The number of chains we denote as n_c and sum of their lengths as N_c.
2. All dangling nodes.
3. All bridges (their number is N_b) and 2-edge cuts, that are not pairs of edges, belonging to a same chain (their number is C_2).
4. All articulation points.
5. The same information for graph G', that is derived from G by removing all chains.

Our task

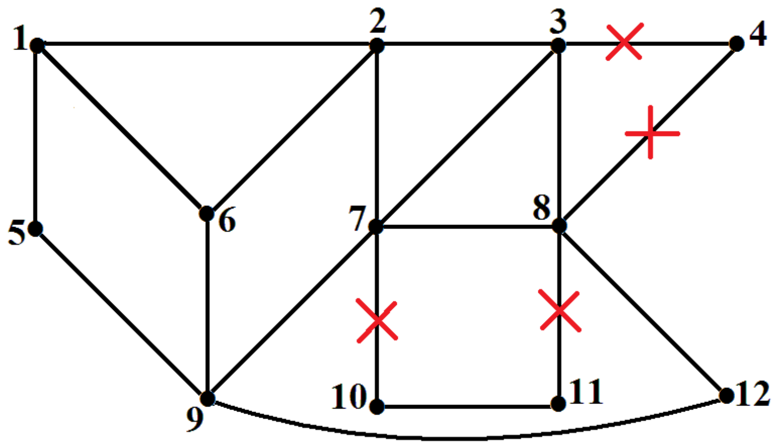
As the task is NP-hard, for large graphs **fast** algorithms are needed for estimating coefficients of reliability polynomial (RP).
As total number of possible graph's destructions is huge, the rough estimations of coefficients may be obtained only. We need counting all trivially known connected and disconnected realizations (states) of a random graph, thus obtain b_i directly and u_i by subtracting number of trivially known disconnected realizations from their total number.

Which graphs are trivially known to be connected or disconnected?

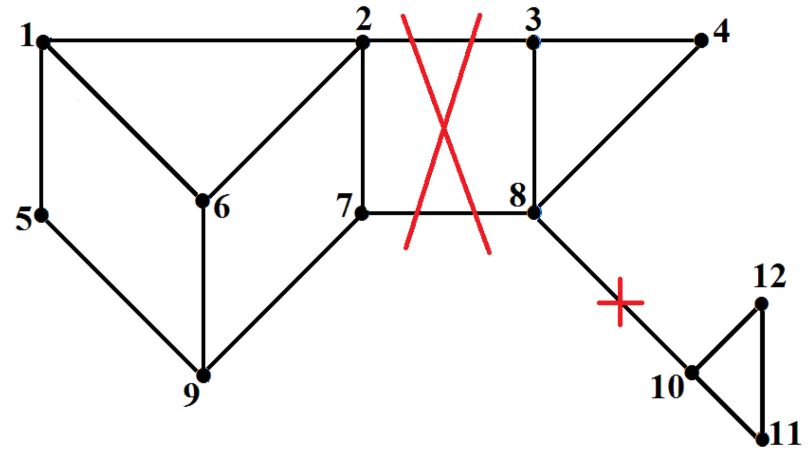
1) Removal of an edge from any chain that is not bridge, cannot destroy graph.



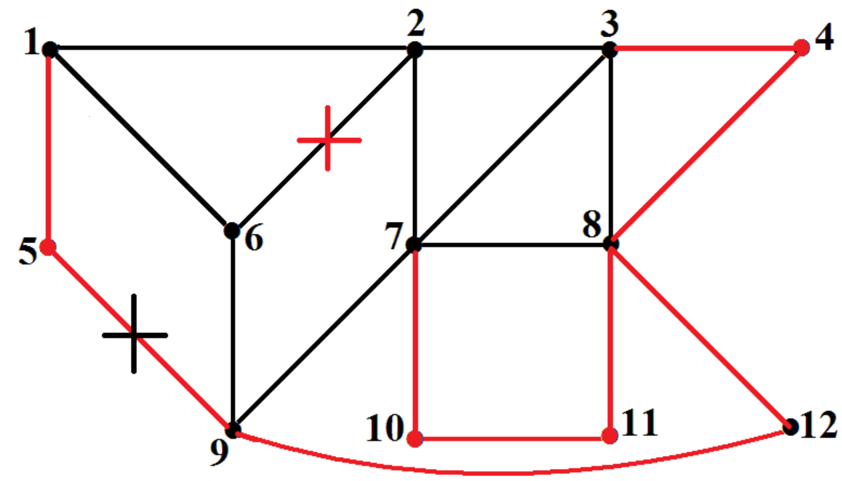
2) Removal of more than one edge from any chain destroys graph for sure.



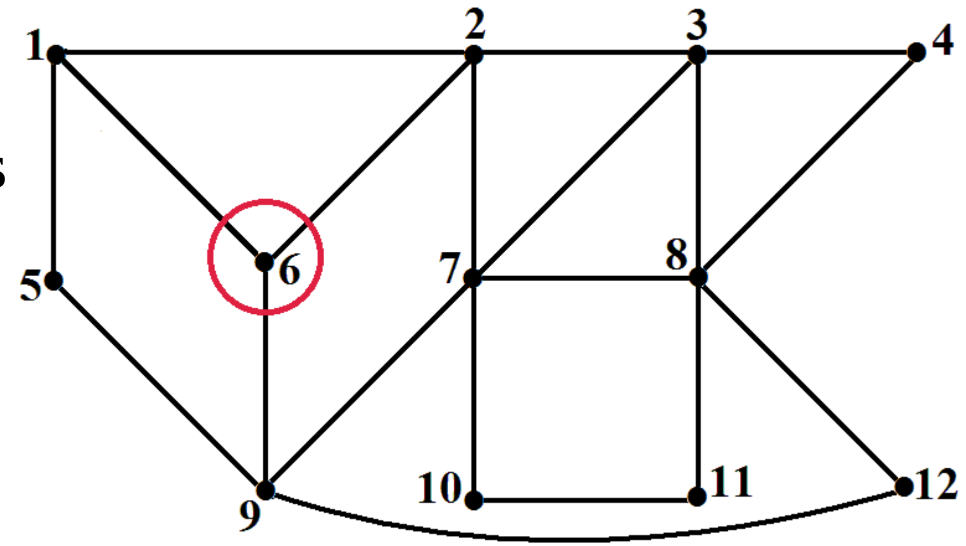
3) Removing any set of edges, that contains a bridge or 2-edge cut will destroy a graph.



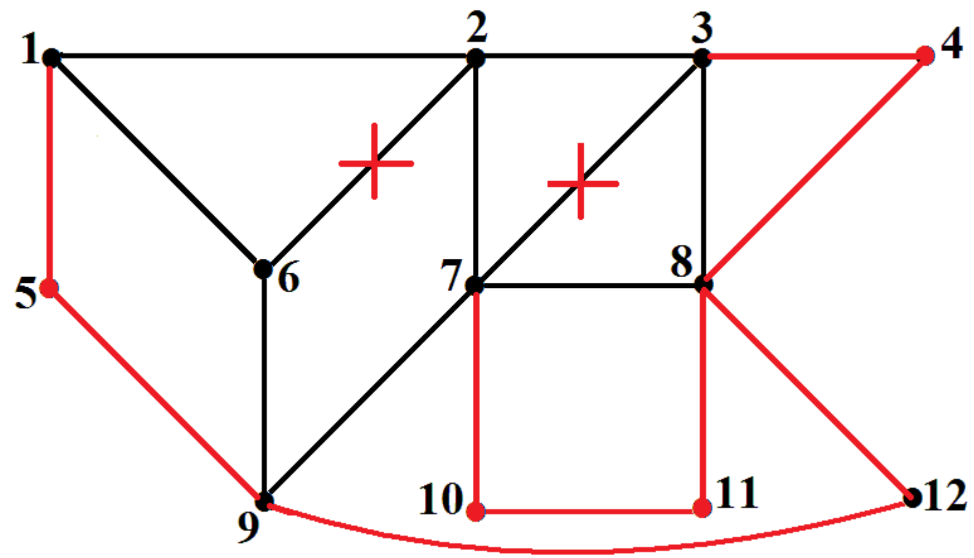
4) Removing an edge from a chain in G and any edge in G' cannot destroy a graph, if they are not bridges in these graphs.



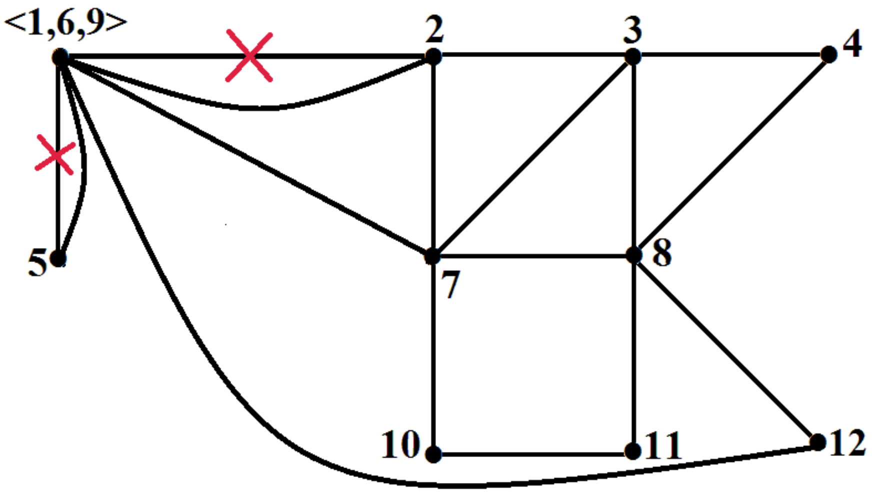
5) Removing all edges incidental to one node destroys a graph.



6) Removing up to 2 edges from *G*’, that does not form a cut in it, does not destroy a graph



7) Removal of any $k < \lambda$ edges from a multi-edge of capacity does nod destroy a graph.



Equations

According to our reasoning:

$$a_i \geq \sum_{U \in N(i,n_c)} \prod_{s \in U} l_s$$

$$a_i \leq C_m^i - \sum_{j=1}^{n_c} \sum_{k=2}^{l_j} C_{l_j}^k C_{m-l_j}^{i-k} - \sum_{j=3}^{\min(MD,i)} |D_j^*| C_{m-N_c-j}^{i-j}$$

Partial cases: $a_0=b_0=u_0=1;$
$a_1=b_1=u_1=m;$

$$a_2 \leq C_m^2 - \sum_{j=1}^{n_c} C_{l_j}^2 - |C_2|$$

$$a_2 \geq C_{m-N_c}^2 + N_c(m-N_c) + \sum_{i=1}^{n_c} l_i(m-l_i) -$$

$$\sum_{i=1}^{n_c-1} \sum_{j=i+1}^{n_c} l_i l_j + 9 * C_{|D_3^*|}^2 - |C_2|,$$

$D_3^* \subseteq D_3$ – a set of non-incident nodes of degree 3, that are not ends of the same chain

$$a_3 \geq b_3 = \sum_{i=1}^{n_c-1} \sum_{j=i+1}^{n_c} l_i l_j (m - N_c - |D_1(G')|);$$

$$a_3 \leq u_3 = C_m^3 - \sum_{k=2}^{3} \sum_{j=1}^{n_c} C_{l_j}^k C_{m-l_j}^{i-k} - |D_1(G')|.$$

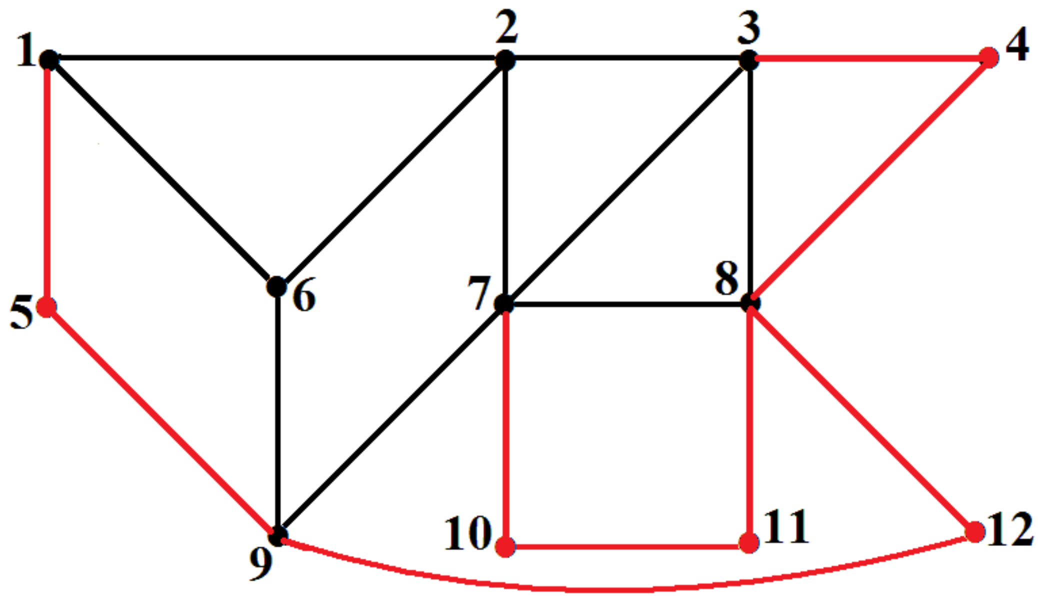
If we use the information about node degrees and existence of chains only, and do not know lengthes of Ch_{1k} and Ch_{2k}, then we may assume that $l_1 = l_2 = 2$,and thus the improved upper bound for a_3 is

$$u_3^* = u_3 - 4|D_1(G')|.$$

If we know lengthes of Ch_{1k} and Ch_{2k} for each k, then we obtain even better upper bound:

$$u_3^{**} = u_3^* - \sum_{k=1}^{|D_1(G')|} l_{1k} l_{2k}.$$

Example



G(12,19), 4 chains (3 with length 2: 1-5-9, 3-4-8, 8-12-9, and one 3: 7-10-11-8), N_c=9.

G'(7,10), 2 chains with length 2, that forms two 2-edge cuts.

$$R(p)=p^{19}+19p^{18}q+165p^{17}q^2+866p^{16}q^3+3043p^{15}q^4+7415p^{14}q^5+12393p^{13}q^6+13212p^{12}q^7+6990p^{11}q^8$$

Direct estimations

a_0=1, a_1=19 and a_8=6990 are known (G is connected, no bridges, number of covering trees is calculated separately).

$$b_2 = C_3^2 \times 2^2 < \text{pairs of edges in chains with lengths 2}> +$$

$$3\times 3\times 2 < \text{pairs of edges in chains with lengths 2 and lengths 3} > +$$

$$9\times 10 < \text{pairs "edge in chain} - \text{edge not in chain"} > +$$

$$C_{10}^2 < \text{pairs of edges in } G' > = 165;$$

$$u_2 = C_{19}^2 - 3 < \text{chains with lengths 2} > -$$

$$C_3^2 < \text{pairs of edges in chains with lengths 2} > = 165.$$

$$b_3 = 2^3 < \text{triples of edges in chains with lengths 2} > +$$

$$3\times 3\times 2^2 < \text{triple of edges in chains, two with lengths 2 and one with lengths 3} > +$$

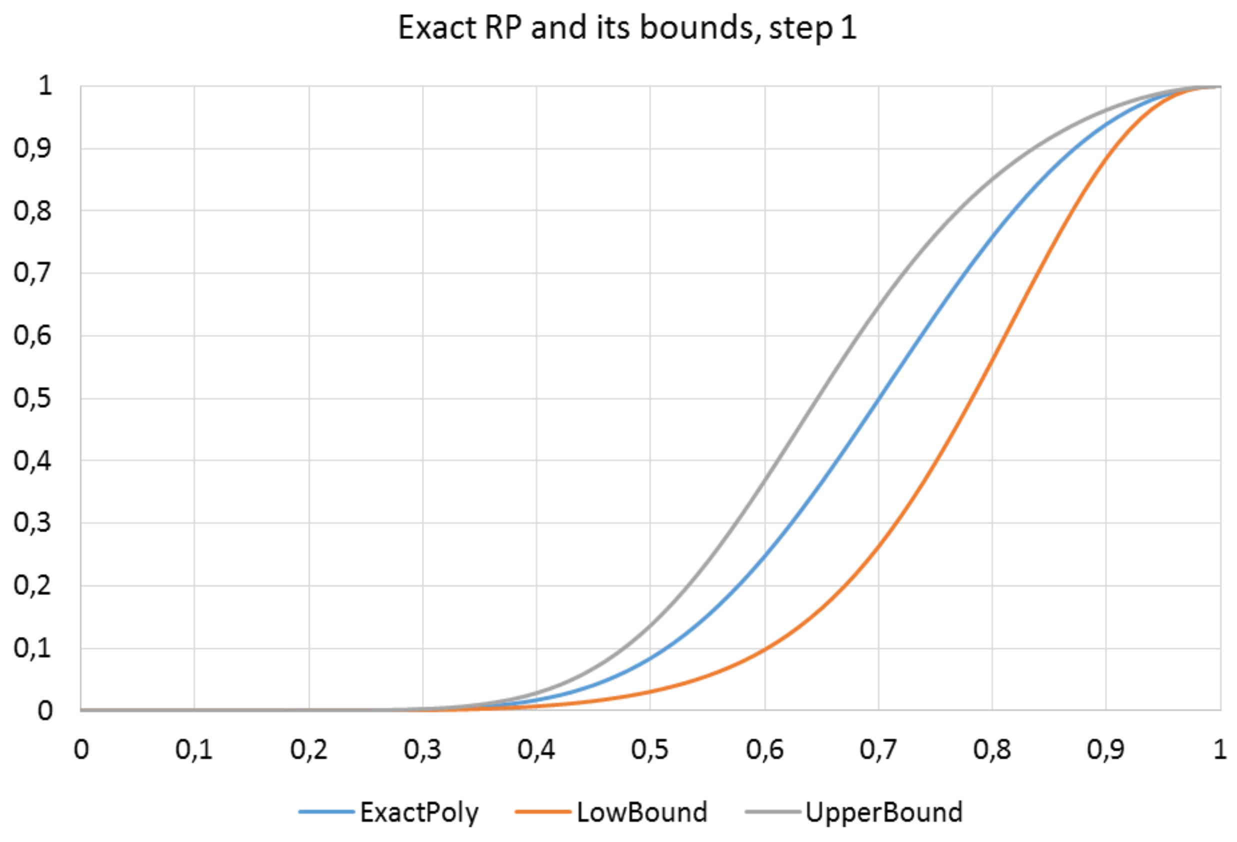
$$10\times(C_3^2 \times 2^2 + 3\times 3\times 2) < \text{pair of edges from different chains and one from } G' > +$$

$$9(\times C_{10}^2 \text{ -1}) < \text{one edge from chains and pair from } G', \text{cut excluded} > = 758;$$

$$R(p) = p^{19}+19p^{18}q+165p^{17}q^2+866p^{16}q^3+3043p^{15}q^4+7415p^{14}q^5 + 12393p^{13}q^6+13212p^{12}q^7+6990p^{11}q^8$$

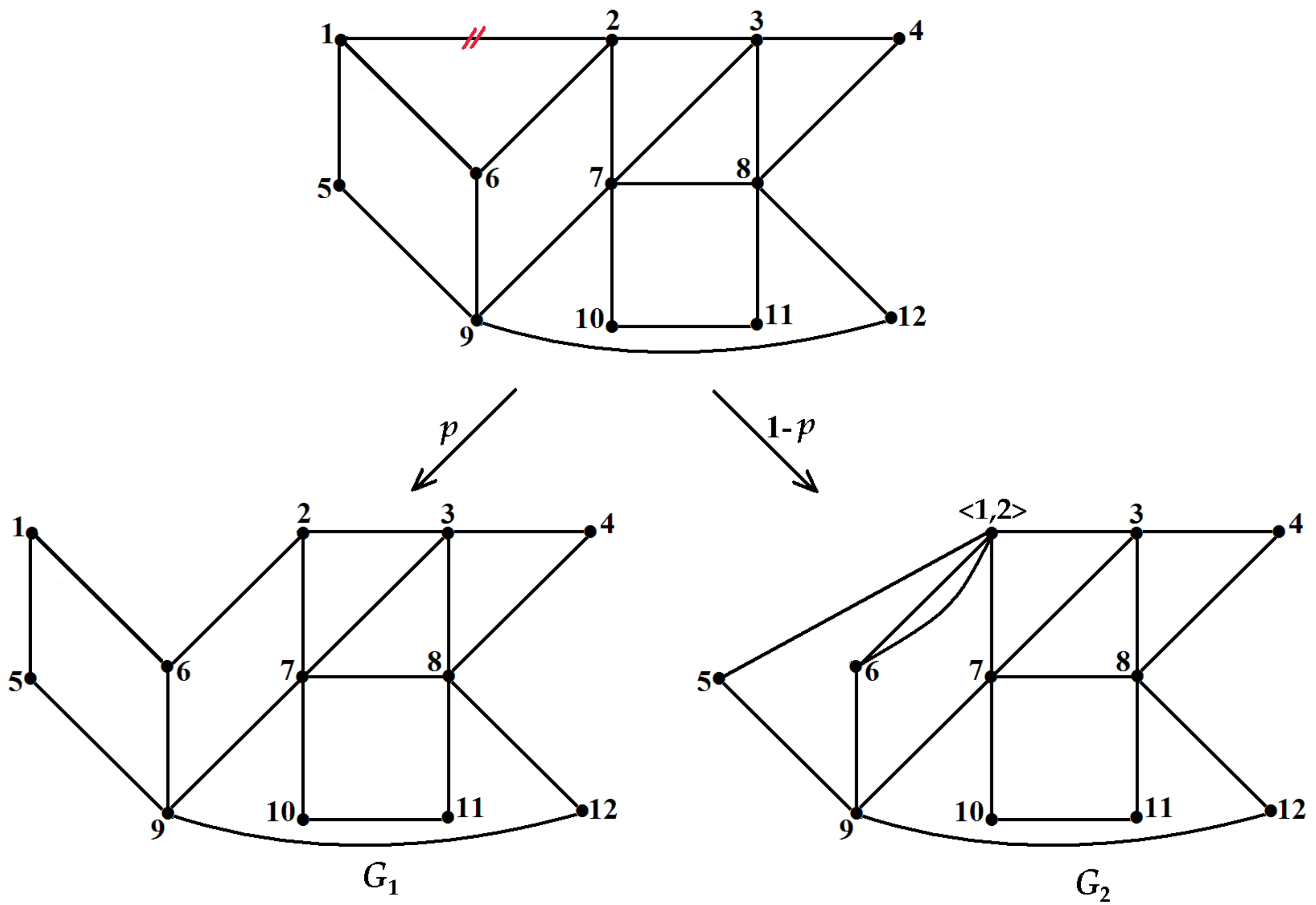
$$B(p) = p^{19} + 19p^{18}q + 165p^{17}q^2 + 758p^{16}q^3 + 1754p^{15}q^4 +216p^{14}q^5 + 28p^{13}q^6 + 8p^{12}q^7 + 6990p^{11}q^8;$$

$$U(p) = p^{19} + 19p^{18}q + 165p^{17}q^2 + 867p^{16}q^3 + 3206p^{15}q^4 +10205p^{14}q^5 + 13677p^{13}q^6 + 25730p^{12}q^7 + 6990p^{11}q^8.$$



May we improve this result?

What if we will make some steps of the factoring process?



We have $R(G)=(1-p)R(G_1)+pR(G_2)$, from which we obtain that $a_i=a_{1,i+1}+a_{2i}$.

If we continue with factoring and examine all variants of states of 2 edges, then

$R(G)=(1-p)^2R(G\backslash e_1\backslash e_2)+(1-p)p[R(G\backslash e_1/e_2)+R(G/e_1\backslash e_2)]+p^2R(G/e_1/e_2)$.

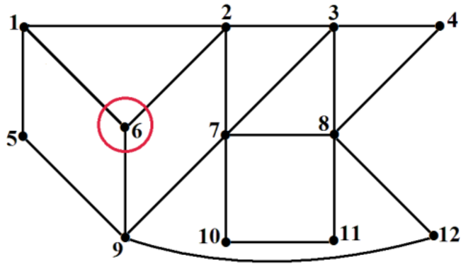
If we continue with factoring and examine all variants of states of 3 edges, then

$R(G)=(1-p)^3R(G\backslash e_1\backslash e_2 \backslash e_3)+(1-p)^2p[R(G\backslash e_1\backslash e_2/e_3)+R(G\backslash e_1/e_2\backslash e_3)+ R(G/e_1\backslash e_2\backslash e_3)]+(1-p)\,p^2[R(G\backslash e_1/e_2/e_3)+R(G/e_1\backslash e_2/e_3)+ R(G/e_1/e_2\backslash e_3)]+ p^3R(G/e_1/e_2/e_3)$.

And so on. As we know exact values of some coefficients for each partial polynomial, quality of estimation increases.

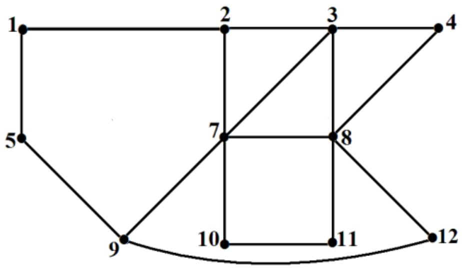
$$R(G,p)=\sum_{i=0}^{m-3} a_{1i}p^{m-i+3}q^i+\sum_{i=0}^{m-3}(a_{2i}+a_{3i}+a_{4i})\,p^{m-i+2}q^i+$$
$$\sum_{i=0}^{m-3}(a_{5i}+a_{6i}+a_{7i})\,p^{m-i+1}q^i+\sum_{i=0}^{m-3}a_{8i}p^{m-i}q^i$$
$$=a_{10}p^m+(a_{11}+a_{20}+a_{30}+a_{40})p^{m-1}q+(a_{12}+a_{21}+a_{31}+a_{41}+a_{50}+a_{60}+a_{70})p^{m-2}q^2+$$
$$\sum_{i=3}^{m-3}(a_{1i}+a_{2,i-1}+a_{3,i-1}+a_{4,i-1}+a_{5,i-2}+a_{6,i-2}+a_{7,i-2}+a_{8,i-3})\,p^{m-i}q^i+$$
$$(a_{2,m-4}+a_{3,m-4}+a_{4,m-4}+a_{5,m-4}+a_{6,m-4}+a_{7,m-4}+a_{8,m-5})\,pq^{m-2}$$
$$(a_{5,m-3}+a_{6,m-3}+a_{7,m-3}+a_{8,m-4})\,pq^{m-1}+a_{8,m-3}q^m.$$

Special case: Factoring by a node with degree 3



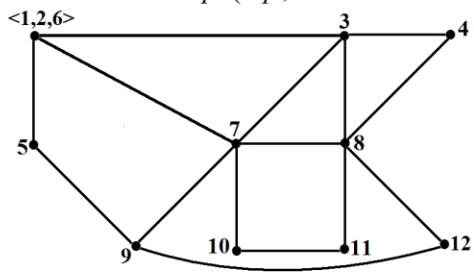
$G(12,19)$

$p(1\text{-}p)^2$



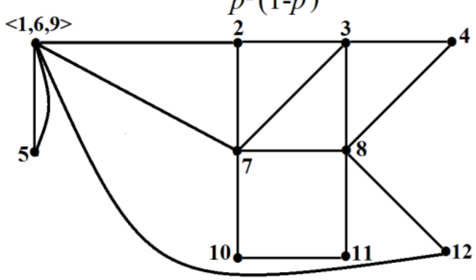
$G_1(11,16)$

$p^2(1\text{-}p)$



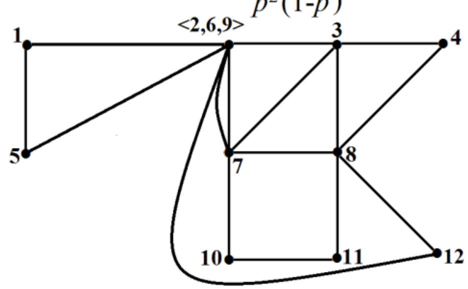
$G_2(10,15)$

$p^2(1\text{-}p)$



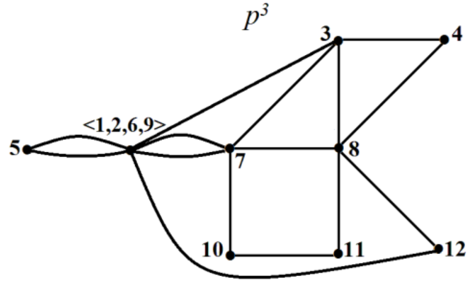
$G_3(10,16)$

$p^2(1\text{-}p)$

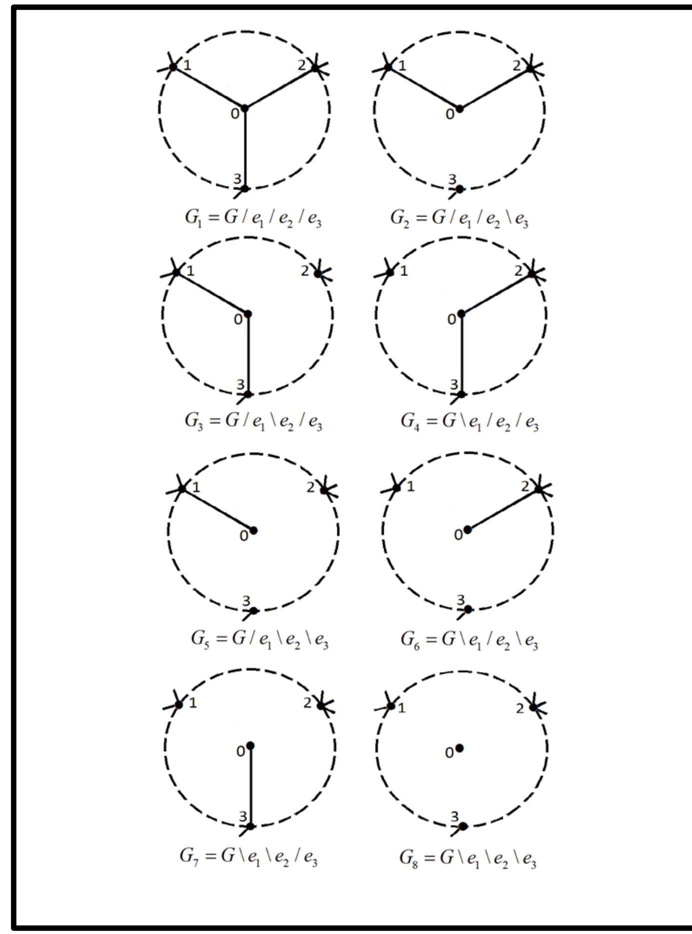


$G_4(10,16)$

p^3



$G_5(9,15)$



Improved polynomials

$$B(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+783p^{16}q^3+1764p^{15}q^4+3194p^{14}q^5+2124p^{13}q^6+936p^{12}q^7+6990p^{11}q^8.$$

$$U(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+933p^{16}q^3+3211p^{15}q^4+9447p^{14}q^5+18864p^{13}q^6+31956p^{12}q^7+6990p^{11}q^8.$$

$$B'(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+846p^{16}q^3+2736p^{15}q^4+5650p^{14}q^5+6744p^{13}q^6+6051p^{12}q^7+6990p^{11}q^8.$$

$$U'(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+867p^{16}q^3+3120p^{15}q^4+8103p^{14}q^5+15490p^{13}q^6+19398p^{12}q^7+6990p^{11}q^8.$$

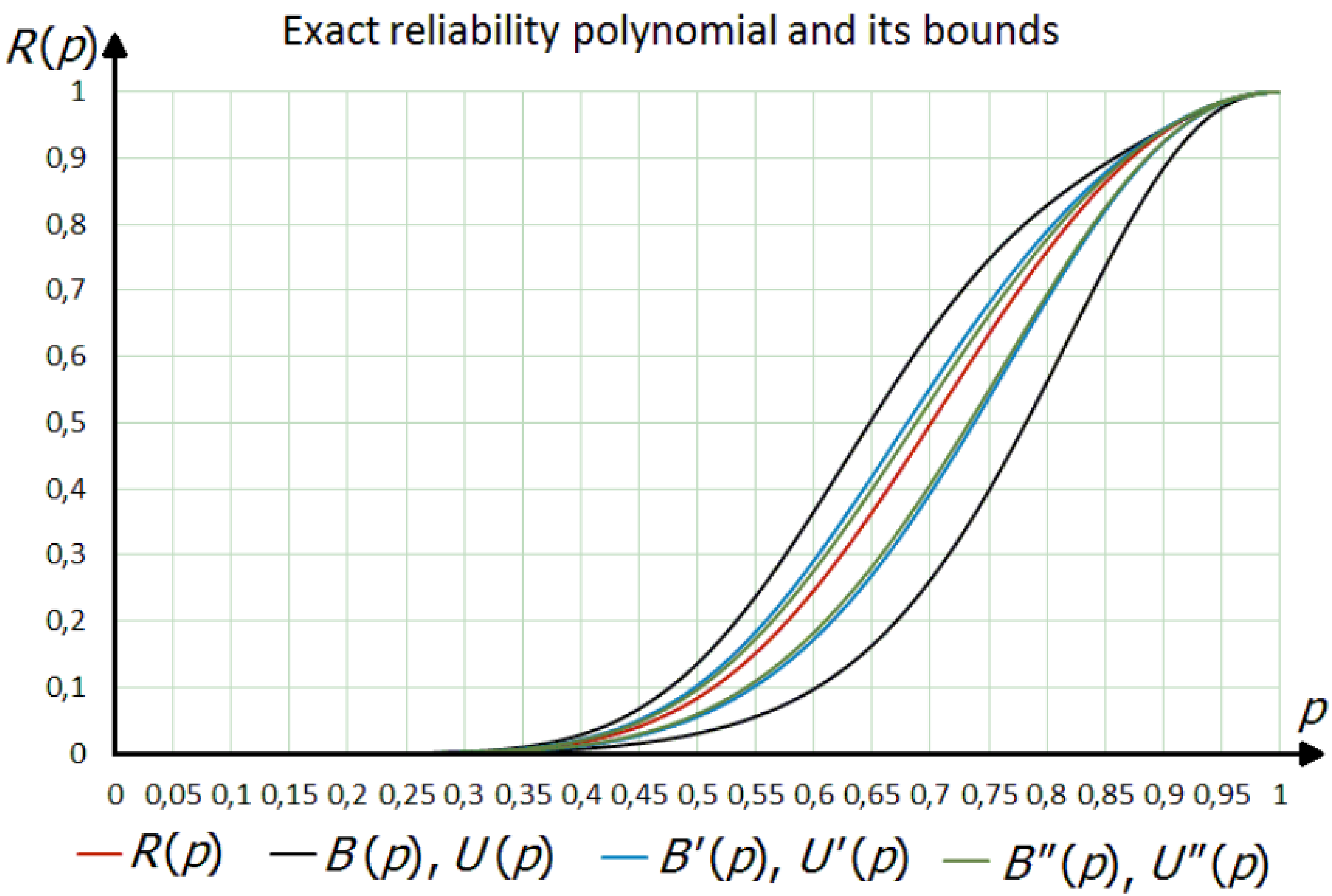
If use decomposition…

$$B'(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+846p^{16}q^3+2736p^{15}q^4+5650p^{14}q^5+6744p^{13}q^6+6051p^{12}q^7+6990p^{11}q^8.$$

$$U'(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+867p^{16}q^3+3120p^{15}q^4+8103p^{14}q^5+15490p^{13}q^6+19398p^{12}q^7+6990p^{11}q^8.$$

$$B''(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+846p^{16}q^3+2765p^{15}q^4+5824p^{14}q^5+7507p^{13}q^6+7285p^{12}q^7+6990p^{11}q^8;$$

$$U''(G,p)=p^{19}+19p^{18}q+165p^{17}q^2+866p^{16}q^3+3119p^{15}q^4+7703p^{14}q^5+14123p^{13}q^6+18047p^{12}q^7+6990p^{11}q^8.$$



Exact reliability polynomial and its bounds
R(p)
1
0,9
0,8
0,7
0,6
0,5
0,4
0,3
0,2
0,1
0
p
0 0,05 0,1 0,15 0,2 0,25 0,3 0,35 0,4 0,45 0,5 0,55 0,6 0,65 0,7 0,75 0,8 0,85 0,9 0,95 1
R(p)
B(p), U(p)
B′(p), U′(p)
B″(p), U″(p)

Conclusion

1) It is clear to see, that majorizing polynomial is closer to exact one than minorizing. It is due to simpler finding possible destructions of a graph, then counting sugraphs, that are connected for sure.
2) Effect of multi-level factoring highly depends on choice of pivot elements.

Show is over

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