Error estimates with explicit constants for Sinc quadrature and Sinc indefinite integration over infinite intervals

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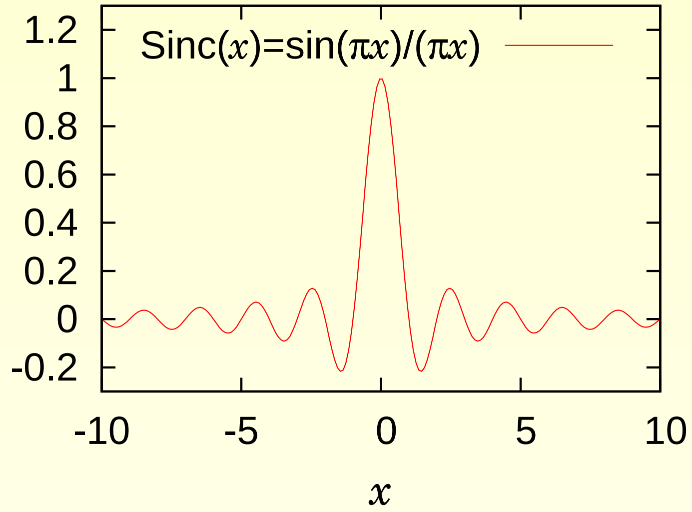
1 Sinc quadrature and Sinc indefinite integration

Base: Sinc approximation on $\mathbb{R}$

$$F(x) \approx \sum_{j=-N}^{N} F(jh)\mathrm{Sinc}(x/h - j)$$

$$\mathrm{Sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$



By integrating both sides, we can derive approximations for

- definite integrals (Sinc quadrature)

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x \approx \sum_{j=-N}^{N} F(jh) \overbrace{\int_{-\infty}^{\infty} \mathrm{Sinc}(x/h - j)\,\mathrm{d}x}^{h}$$

- indefinite integrals (Sinc indefinite integration)

$$\int_{-\infty}^{\tau} F(x)\,\mathrm{d}x \approx \sum_{j=-N}^{N} F(jh) \int_{-\infty}^{\tau} \mathrm{Sinc}(x/h - j)\,\mathrm{d}x$$

Important facts on two-step approximation

Sinc quadrature $\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x \approx h \sum_{j=-N}^{N} F(jh)$

$$\int_{-\infty}^{\infty} F(x)\,\mathrm{d}x \approx h \sum_{j=-\infty}^{\infty} F(jh)$$ discretization

(accurate if the integral is over $\mathbb{R}$)

$$\approx h \sum_{j=-N}^{N} F(jh)$$ truncation

(accurate if $|F(x)|$ decays quickly as $x \to \pm\infty$)

Question When either the two "if" is NG ...?

Four typical cases that Stenger [2] considered

Let f be a smooth and bounded function

1. $I_1 = (-\infty, \infty)$ $\displaystyle\int_{-\infty}^{\infty} \frac{f(t)}{1+t^2}\,\mathrm{d}t$ polynomial decay

2. $I_2 = (0, \infty)$ $\displaystyle\int_{0}^{\infty} \frac{f(t)}{1+t^2}\,\mathrm{d}t$ polynomial decay

3. $I_3 = (0, \infty)$ $\displaystyle\int_{0}^{\infty} \mathrm{e}^{-t} f(t)\,\mathrm{d}t$ exponential decay

4. $I_4 = (a, b)$ $\displaystyle\int_{a}^{b} f(t)\,\mathrm{d}t$ (not decay)

Variable transformations for the 4 cases

Single-Exponential transformation $x = \psi_{\mathsf{SE}i}(x)$ (Stenger [2])

$$I_1 = (-\infty, \infty) \qquad F(t) = \frac{1}{1+t^2} f(t)$$

$$I_2 = (0, \infty) \qquad F(t) = \frac{1}{1+t^2} f(t)$$

$$I_3 = (0, \infty) \qquad F(t) = \mathrm{e}^{-t} f(t)$$

$$I_4 = (a, b) \qquad F(t) = f(t)$$

Variable transformations for the 4 cases

Single-Exponential transformation $x = \psi_{\mathsf{SE}i}(x)$ (Stenger [2])

$$I_1 = (-\infty, \infty) \qquad F(t) = \frac{1}{1+t^2} f(t)$$

$$t = \psi_{\mathsf{SE1}}(x) = \sinh x$$

$$I_2 = (0, \infty) \qquad F(t) = \frac{1}{1+t^2} f(t)$$

$$t = \psi_{\mathsf{SE2}}(x) = \mathrm{e}^x$$

$$I_3 = (0, \infty) \qquad F(t) = \mathrm{e}^{-t} f(t)$$

$$t = \psi_{\mathsf{SE3}}(x) = \operatorname{arcsinh}(\mathrm{e}^x)$$

$$I_4 = (a, b) \qquad F(t) = f(t)$$

$$t = \psi_{\mathsf{SE4}}(x) = \tfrac{b-a}{2} \tanh\left(\tfrac{x}{2}\right) + \tfrac{b+a}{2}$$

SE-Sinc quadrature (case 2)

$$\int_{I_2} F(t)\,\mathrm{d}t = \int_{-\infty}^{\infty} \underline{F(\psi_{\mathsf{SE2}}(x))\psi'_{\mathsf{SE2}}(x)}\,\mathrm{d}x$$ SE transformation

$$\approx h \sum_{j=-\infty}^{\infty} F(\psi_{\mathsf{SE2}}(jh))\psi'_{\mathsf{SE2}}(jh)$$ discretization

(accurate because the integral is over $\mathbb{R}$)

$$\approx h \sum_{j=-N}^{N} F(\psi_{\mathsf{SE2}}(jh))\psi'_{\mathsf{SE2}}(jh)$$ truncation

(accurate because the integrand decays $\mathrm{O}(\mathrm{e}^{-|x|})$)

To make the truncation error further smaller

Double-Exponential transformation $x = \psi_{\mathrm{DE}i}(x)$ (Takahasi–Mori [3])

$$I_1 = (-\infty, \infty) \qquad F(t) = \frac{1}{1+t^2} f(t)$$

$$t = \psi_{\mathrm{DE1}}(x) = \sinh[(\pi/2)\sinh x]$$

$$I_2 = (0, \infty) \qquad F(t) = \frac{1}{1+t^2} f(t)$$

$$t = \psi_{\mathrm{DE2}}(x) = \mathrm{e}^{(\pi/2)\sinh x}$$

$$I_3 = (0, \infty) \qquad F(t) = \mathrm{e}^{-t} f(t)$$

$$t = \psi_{\mathrm{DE3}}(x) = \mathrm{e}^{x - \exp(-x)}$$

$$I_4 = (a, b) \qquad F(t) = f(t)$$

$$t = \psi_{\mathrm{DE4}}(x) = \tfrac{b-a}{2}\tanh\left(\tfrac{\pi \sinh x}{2}\right) + \tfrac{b+a}{2}$$

Decay rate: $\mathrm{O}(\mathrm{e}^{-\exp(|x|)})$

Error analysis for SE/DE-Sinc quadrature

Case 1 $$\int_{-\infty}^{\infty} \frac{\sqrt{1+\tanh^2(\operatorname{arcsinh}(t)/2)}}{1+t^2}\,\mathrm{d}t = 4\operatorname{arcsinh}(1)$$

There exists a constant C independent of N such that

$|\text{Error (SE)}| \leq C\,\mathrm{e}^{-\sqrt{\pi^2 N/2}}$ (Stenger [2])

$|\text{Error (DE)}| \leq C\,\mathrm{e}^{-(\pi^2 N/2)/\log(4\pi N)}$ (Tanaka et al. [4])

Convergence rates are given (C is not estimated explicitly)

Contribution 1. Explicit estimates of the C's

Case 1 $$\int_{-\infty}^{\infty} \frac{\sqrt{1+\tanh^2(\operatorname{arcsinh}(t)/2)}}{1+t^2}\,\mathrm{d}t = 4\operatorname{arcsinh}(1)$$

There exists a constant C independent of N such that

$$|\text{Error (SE)}| \leq C\,\mathrm{e}^{-\sqrt{\pi^2 N/2}} \qquad \text{(Stenger [2])}$$

where $C = 122.6$ (*New!*)

$$|\text{Error (DE)}| \leq C\,\mathrm{e}^{-(\pi^2 N/2)/\log(4\pi N)} \qquad \text{(Tanaka et al. [4])}$$

where $C = 2345$ (*New!*)

Now the errors can be estimated by computing the R.H.S.!

Summary table for the explicit estimates

SE/DE-Sinc quadrature	Case 1	$(-\infty, \infty)$	*New!*
	Case 2	$(0, \infty)$	*New!*
	Case 3	$(0, \infty)$	*New!*
	Case 4	(a, b)	OK [1]
SE/DE-Sinc indefinite integration	Case 1	$(-\infty, \infty)$	*New!*
	Case 2	$(0, \infty)$	*New!*
	Case 3	$(0, \infty)$	*New!*
	Case 4	(a, b)	OK [1]

- The desired explicit estimates are already done in Case 4 [1]
- This study extends the result to Case 1–3

Issue on the DE transformation in Case 3

- Inverse of $t = \psi_{\mathrm{DE3}}(x) = \mathrm{e}^{x - \exp(-x)}$ cannot be written explicitly (in elementary functions)

- For the Sinc indefinite integration, its inverse is needed (not needed for the Sinc quadrature)

- For the purpose, Muhammad–Mori (2003) proposed
  $t = \psi_{\mathrm{DE3}\dagger}(x) = \log(1 + \mathrm{e}^{(\pi/2)\sinh x})$

Contribution 2: improving the transformation

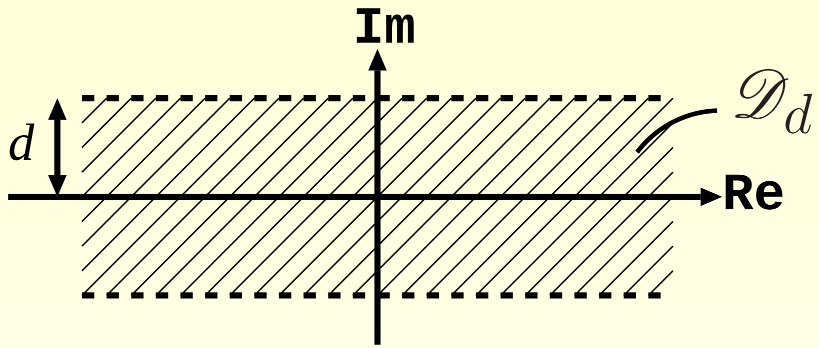
- Inverse of $t = \psi_{\text{DE3}}(x) = \mathrm{e}^{x-\exp(-x)}$ cannot be written explicitly (in elementary functions)
- For the Sinc indefinite integration, its inverse is needed (not needed for the Sinc quadrature)
- For the purpose, Muhammad–Mori (2003) proposed $t = \psi_{\text{DE3}\dagger}(x) = \log(1 + \mathrm{e}^{(\pi/2)\sinh x})$
- In this talk, the transformation is improved as $t = \psi_{\text{DE3}\ddagger}(x) = \log(1 + \mathrm{e}^{\pi\sinh x})$ (and this is optimal)

Outline of this talk

2 Improving the DE transformation in Case 3

Definition: the strip domain $\mathscr{D}_d$

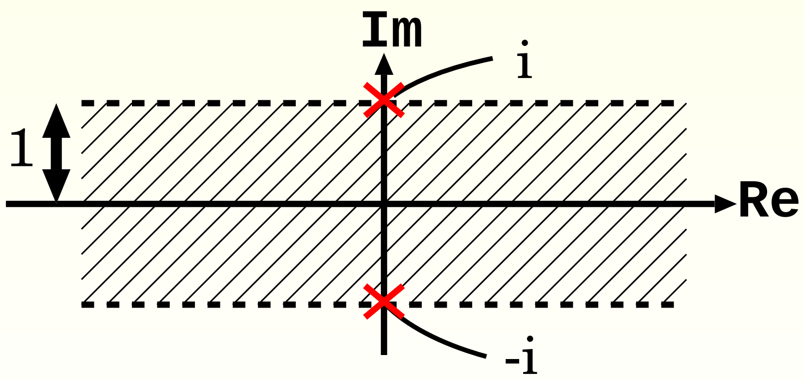
The integrand should be analytic on the complex domain $\mathscr{D}_d$:



$$\mathscr{D}_d = \{z \in \mathbb{C} : |\operatorname{Im} z| < d\}$$

$d > 0$

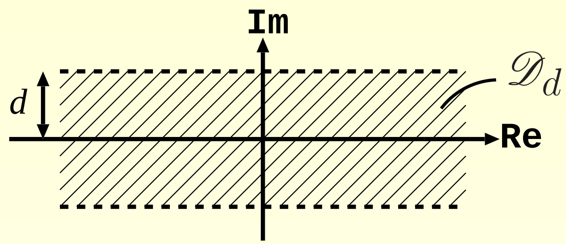
Example $f(x) = \dfrac{1}{1+x^2}$ is analytic on $\mathscr{D}_1$



Sketch of the existing error analysis [4]

Point discretization error E_{D} and truncation error E_{T}

Let $f(\psi_{\mathsf{DE3}\dagger}(\cdot))$ be analytic and bounded on $\mathscr{D}_d$



Integrand $F_{\mathsf{DE3}\dagger}(x) = \mathrm{e}^{-\psi_{\mathsf{DE3}\dagger}(x)} f(\psi_{\mathsf{DE3}\dagger}(x))\psi'_{\mathsf{DE3}\dagger}(x)$

$$\int_0^\infty \mathrm{e}^{-t} f(t)\,\mathrm{d}t \approx h \sum_{k=-\infty}^{\infty} F_{\mathsf{DE3}\dagger}(kh) \qquad E_{\mathrm{D}} = \mathrm{O}(\mathrm{e}^{-2\pi d/h})$$

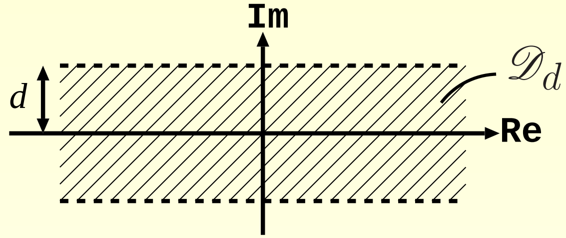
$$\approx h \sum_{k=-N}^{N} F_{\mathsf{DE3}\dagger}(kh) \qquad E_{\mathrm{T}} = \mathrm{O}\left(\mathrm{e}^{-\frac{\pi}{4}\exp(Nh)}\right)$$

$$\frac{2\pi d}{h} \approx \frac{\pi}{4}\mathrm{e}^{Nh} \;\Rightarrow\; h = \frac{\log(8dN)}{N} \qquad E_{\mathsf{DE}} = \mathrm{O}\left(\exp\left\{\frac{-2\pi dN}{\log(8dN)}\right\}\right)$$

Improvement of the DE transformation

Point discretization error E_{D} and truncation error E_{T}

Let $f(\psi_{\mathrm{DE3}\dagger}(\cdot))$ be analytic and bounded on $\mathscr{D}_d$



$$\psi_{\mathrm{DE3}\dagger}(t) = \log(1 + \mathrm{e}^{(\pi/2)\sinh(t)}) \to \psi_{\mathrm{DE3}\ddagger}(t) = \log(1 + \mathrm{e}^{\pi\sinh(t)})$$

$$\int_0^\infty \mathrm{e}^{-t} f(t)\,\mathrm{d}t \approx h \sum_{k=-\infty}^{\infty} F_{\mathrm{DE3}\ddagger}(kh) \qquad E_{\mathrm{D}} = \mathrm{O}(\mathrm{e}^{-2\pi d/h})$$

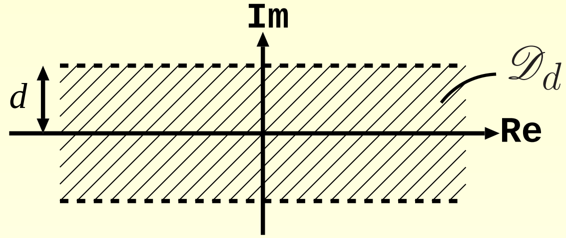
$$\approx h \sum_{k=-N}^{N} F_{\mathrm{DE3}\ddagger}(kh) \qquad E_{\mathrm{T}} = \mathrm{O}\left(\mathrm{e}^{-\frac{\pi}{2}\exp(Nh)}\right)$$

$$\frac{2\pi d}{h} \approx \frac{\pi}{2}\mathrm{e}^{Nh} \Rightarrow h = \frac{\log(4dN)}{N} \qquad E_{\mathrm{DE}} = \mathrm{O}\left(\exp\left\{\frac{-2\pi d N}{\log(4dN)}\right\}\right)$$

Is further improvement possible?

Point: discretization error E_{D} and truncation error E_{T}

Let $f(\psi_{\mathsf{DE3}\dagger}(\cdot))$ be analytic and bounded on $\mathscr{D}_d$



$$\psi_{\mathsf{DE3}\ddagger}(t) = \log(1 + \mathrm{e}^{\pi \sinh(t)}) \to \psi_{\mathsf{DE3}*}(t) = \log(1 + \mathrm{e}^{2\pi \sinh(t)})$$

$$\int_0^{\infty} \mathrm{e}^{-t} f(t)\,\mathrm{d}t \approx h \sum_{k=-\infty}^{\infty} F_{\mathsf{DE3}*}(kh) \qquad E_{\mathrm{D}} = \mathrm{O}(\mathrm{e}^{-2\pi d/h})$$

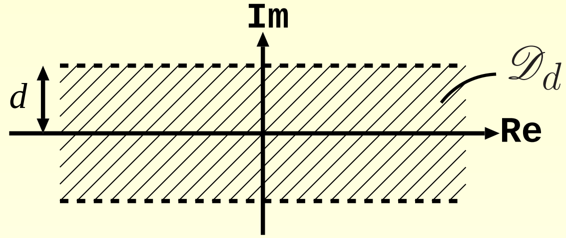
$$\approx h \sum_{k=-N}^{N} F_{\mathsf{DE3}*}(kh) \qquad E_{\mathrm{T}} = \mathrm{O}\left(\mathrm{e}^{-\frac{\pi}{1}\exp(Nh)}\right)$$

$$\frac{2\pi d}{h} \approx \frac{\pi}{1}\mathrm{e}^{Nh} \Rightarrow h = \frac{\log(2dN)}{N} \qquad E_{\mathsf{DE}} = \mathrm{O}\left(\exp\left\{\frac{-2\pi dN}{\log(2dN)}\right\}\right)$$

Not possible (Sugihara 1998)

Point discretization error E_{D} and truncation error E_{T}

Let $f(\psi_{\mathsf{DE3}\dagger}(\cdot))$ be analytic and bounded on $\mathscr{D}_d$



$$\psi_{\mathsf{DE3}\ddagger}(t) = \log(1 + \mathrm{e}^{\pi \sinh(t)}) \to \psi_{\mathsf{DE3}*}(t) = \log(1 + \mathrm{e}^{2\pi \sinh(t)})$$

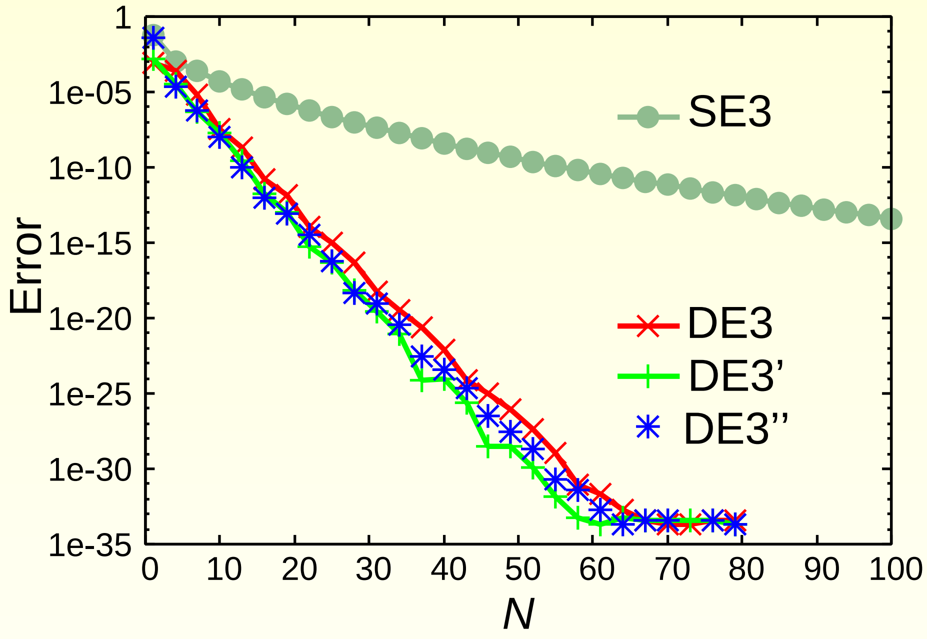
$$\int_0^\infty \mathrm{e}^{-t} f(t)\,\mathrm{d}t \approx h \sum_{k=-\infty}^{\infty} F_{\mathsf{DE3}*}(kh) \qquad E_{\mathrm{D}} = \mathrm{O}(\mathrm{e}^{-2\pi d/h})$$

$$\approx h \sum_{k=-N}^{N} F_{\mathsf{DE3}*}(kh) \qquad E_{\mathrm{T}} = \mathrm{O}\left(\mathrm{e}^{-\frac{\pi}{1}\exp(Nh)}\right)$$

$$\frac{2\pi d}{h} \approx \frac{\pi}{1}\mathrm{e}^{Nh} \Rightarrow h = \frac{\log(2dN)}{N} \qquad E_{\mathsf{DE}} = \mathrm{O}\left(\exp\left\{\frac{-2\pi dN}{\log(2dN)}\right\}\right)$$

Numerical Example (C, quadruple precision)

$$\int_0^\infty e^{-t}\sqrt{1+\tanh^2(\log(\sinh(t))/2)}\,dt = 4\operatorname{arcsinh}(1) - \sqrt{2}(1+\log 2)$$



- Almost no difference between $\psi_{\text{DE3}}(t)$, $\psi_{\text{DE3}\dagger}(t)$, $\psi_{\text{DE3}\ddagger}(t)$?
- For $\psi_{\text{DE3}\ddagger}(t)$, we have explicit error estimates (next)

3 Error estimates with explicit constants

ToDo: Bound the transformed integrand

Case 1

$$\int_{-\infty}^{\infty} \frac{f(t)}{1+t^2}\,\mathrm{d}t = \int_{-\infty}^{\infty} \frac{f(\psi(x))}{1+\{\psi(x)\}^2}\psi'(x)\,\mathrm{d}x$$

$(\psi(x) = \psi_{\mathsf{SE1}}(x)$ or $\psi_{\mathsf{DE1}}(x))$

- $|f(\psi(z))| \leq K$: Suppose given by users (depends on the problem).
- $\left|\dfrac{\psi'(z)}{1+\{\psi(z)\}^2}\right| \leq ??$: Evaluated by this study (always the same)

Existing bound

$$\left|\frac{\psi_{\mathsf{SE1}}'(z)}{1+\{\psi_{\mathsf{SE1}}(z)\}^2}\right| \leq C\,\mathrm{e}^{-|\operatorname{Re} z|}, \quad \left|\frac{\psi_{\mathsf{DE1}}'(z)}{1+\{\psi_{\mathsf{DE1}}(z)\}^2}\right| \leq C\,\mathrm{e}^{-\exp(|\operatorname{Re} z|)}$$

The explicit form of C's is revealed in this study (for Case 1–3), which enables us to obtain the desired explicit error estimates.

Error estimates with explicit constants (SE)

Theorem

- $f(\psi_{\mathsf{SE}i}(\cdot))$ is analytic on $\mathscr{D}_d$. $(i=1,\,2,\,3)$
- $\forall z \in \mathscr{D}_d$, $|f(\psi_{\mathsf{SE}i}(z)| \leq K$. $(i=1,\,2,\,3)$

$\Longrightarrow$

$$\left|E_{\mathsf{SE}\text{-Sinc quadrature}}\right| \leq K C_{\mathsf{SE}} C_i \,\mathrm{e}^{-\sqrt{2\pi d N}},$$

where

$$C_{\mathsf{SE}} = 1 + \frac{2}{(1-\mathrm{e}^{-\sqrt{2\pi d}})\cos d}, \quad C_1 = 2^2, \quad C_2 = 2, \quad C_3 = 2^{1/2}.$$

Error estimates with explicit constants (DE)

Theorem

- $f(\psi_{\mathrm{DE}i}(\cdot))$ is analytic on $\mathscr{D}_d$. $(i = 1,\, 2,\, 3)$
- $\forall z \in \mathscr{D}_d$, $|f(\psi_{\mathrm{DE}i}(z)| \leq K$. $(i = 1,\, 2,\, 3)$

$\Longrightarrow$

$$\left|E_{\text{DE-Sinc quadrature}}\right| \leq K C_{\mathrm{DE}} C_i \, \mathrm{e}^{-2\pi d N / \log(8dN)}, \quad (i = 1,\, 2)$$

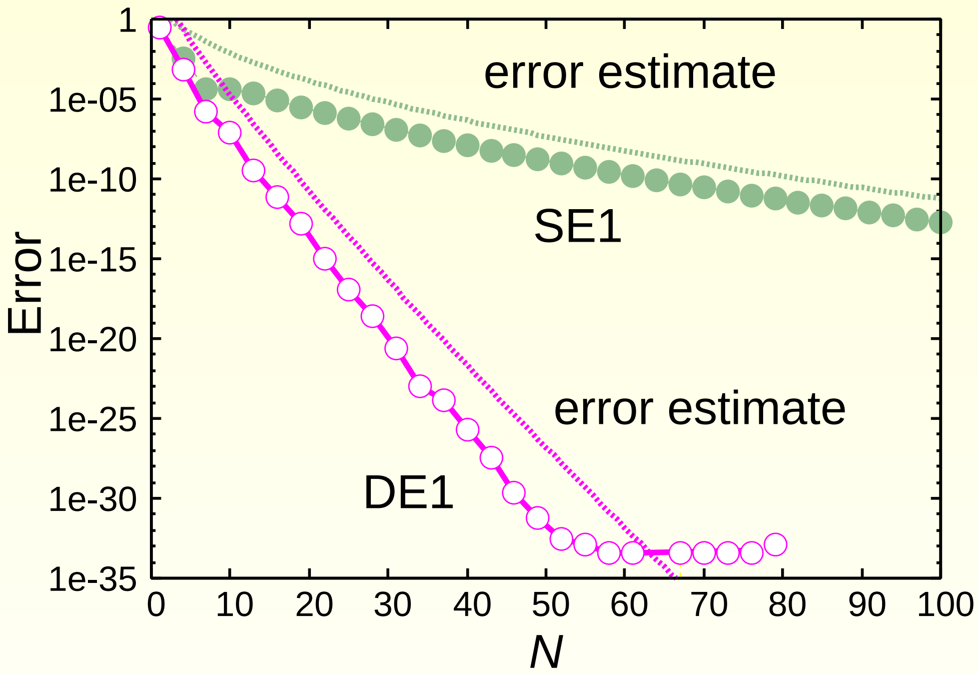
$$\left|E_{\text{DE-Sinc quadrature}}\right| \leq K C'_{\mathrm{DE}} C_i \, \mathrm{e}^{-2\pi d N / \log(4dN)}, \quad (i = 3)$$

where

$$C_{\mathrm{DE}},\, C'_{\mathrm{DE}} = (\text{only depend on } d), \quad C_1 = 2^2, \quad C_2 = 2, \quad C_5 = 2^{1/2}.$$

Numerical Example (C, quadruple precision)

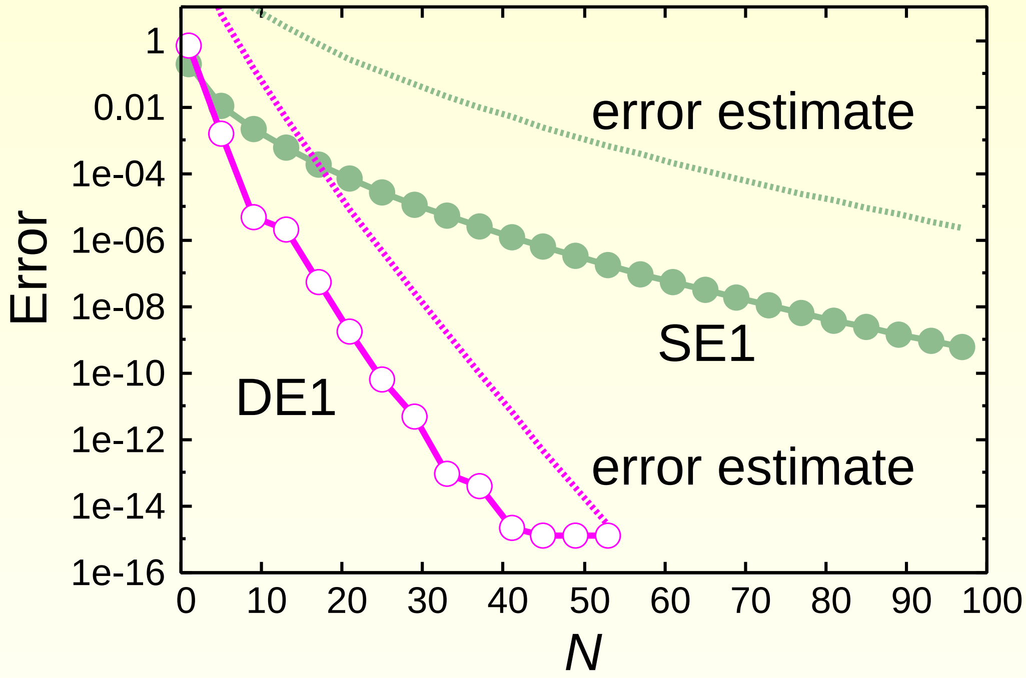
Case 1 $$\int_{-\infty}^{\infty} \frac{\sqrt{1+\tanh^2(\operatorname{arcsinh}(t)/2)}}{1+t^2}\,\mathrm{d}t = 4\operatorname{arcsinh}(1)$$



- "error estimate" bounds the actual error
- Similar results can be obtained in Cases 2 and 3

Numerical Example (C, double precision)

Case 1 $$\int_{-\infty}^{x} \frac{1}{1+t^2}\,\mathrm{d}t = \frac{\pi}{2} + \arctan(x)$$



- "error estimate" bounds the actual error
- Similar results can be obtained in Cases 2 and 3

4 Summary and future work

Error estimates with explicit constants

SE/DE-Sinc quadrature	Case 1	$(-\infty, \infty)$	*New!*
	Case 2	$(0, \infty)$	*New!*
	Case 3	$(0, \infty)$	*New!*
	Case 4	(a, b)	OK [1]
SE/DE-Sinc indefinite integration	Case 1	$(-\infty, \infty)$	*New!*
	Case 2	$(0, \infty)$	*New!*
	Case 3	$(0, \infty)$	*New!*
	Case 4	(a, b)	OK [1]

$t = \psi_{\mathsf{DE3}}(x) = \mathrm{e}^{x-\exp(-x)} \rightarrow t = \psi_{\mathsf{DE3}\dagger}(x) = \log(1 + \mathrm{e}^{(\pi/2)\sinh x})$

$\rightarrow t = \psi_{\mathsf{DE3}\ddagger}(x) = \log(1 + \mathrm{e}^{\pi \sinh x})$ (*New!*)

Future work

SE/DE-Sinc quadrature	Case 1	$(-\infty, \infty)$	*New!*
	Case 2	$(0, \infty)$	*New!*
	Case 3	$(0, \infty)$	*New!*
	Case 4	(a, b)	OK [1]
SE/DE-Sinc indefinite integration	Case 1	$(-\infty, \infty)$	*New!*
	Case 2	$(0, \infty)$	*New!*
	Case 3	$(0, \infty)$	*New!*
	Case 4	(a, b)	OK [1]

- Explicit error estimates for the Sinc approximation (almost done)
- Application to the integral equations over the infinite interval