Model predictive control of discrete linear systems with interval and stochastic uncertainties

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Problem statement

We consider a linear dynamic system described by the following equation:

$$x(k+1) = \left(A_0(k)+\sum_{j=1}^{n} A_j(k)w_j(k)\right)x(k)+\left(B_0(k)+\sum_{j=1}^{n} B_j(k)w_j(k)\right)u(k),$$

$$k = 0, 1, 2, \ldots, \quad (1)$$

where

$x(k) \in \mathbb{R}^{n_x}$ is the state of the system at time k;

$u(k) \in \mathbb{R}^{n_u}$ is the control input at time k;

$w_j(k), j = 1, \ldots, n,$ are independent white noises with zero mean and unit variance;

$A_j(k) \in \mathbb{R}^{n_x \times n_x}$, $B_j(k) \in \mathbb{R}^{n_x \times n_u}$, $j = 0, \ldots, n,$ are the state-space matrices of the system.

The elements of the state-space matrices are known not exactly, and we have only the intervals of their possible values:

$$A_j(k) \in \mathbf{A}_j, \quad B_j(k) \in \mathbf{B}_j, \quad j = 0, \ldots, n, \quad k \geq 0, \tag{2}$$

where

$\mathbf{A}_j \in \mathbb{IR}^{n_x \times n_x}, \mathbf{B}_j \in \mathbb{IR}^{n_x \times n_u}, j = 0, \ldots, n;$

$\mathbb{IR}$ is the set of the real intervals $\mathbf{x} = [\underline{x}, \overline{x}]$, $\underline{x} \leq \overline{x}$, $\underline{x}, \overline{x} \in \mathbb{R}$.

The interval matrices $\mathbf{A}_j, \mathbf{B}_j, j = 0, \ldots, n$, create a polytope

$$\Omega = \mathrm{Co}\left\{[A_{01} \ldots A_{n1} \; B_{01} \ldots B_{n1}], \ldots, [A_{0L} \ldots A_{nL} \; B_{0L} \ldots B_{nL}]\right\},$$

where $\mathrm{Co}\{\cdot\}$ is a convex hull. Then the condition (2) can be described as:

$$[A_0(k) \ldots A_n(k) B_0(k) \ldots B_n(k)] \in \Omega, \quad k \geq 0. \tag{3}$$

We consider the following performance objective:

$$\min_{u(k+i|k)=F(k)x(k+i|k), i\geq 0,} \quad \max_{[A_0(k+i) \ldots A_n(k+i) B_0(k+i) \ldots B_n(k+i)] \in \Omega,\ i\geq 0,} \quad J(k),$$

where

$$J(k) = \mathsf{E}\left\{ \sum_{i=0}^{\infty} \left(x(k+i|k)^T Q x(k+i|k) + u(k+i|k)^T R u(k+i|k) \right) \,\middle|\, x(k) \right\} \tag{4}$$

$\mathsf{E}\{\cdot|\cdot\}$ denotes the conditional expectation;

Q, R are symmetric positive definite weighting matrices, $Q > 0, R > 0$.

$u(k+i|k)$ is the predictive control at time $k+i$ computed at time k, and $u(k|k)$ is the control move implemented at time k;

$x(k+i|k)$ is the state of the system at time $k+i$ derived at time k by applying the sequence of predictive controls $u(k|k), u(k+1|k), \ldots, u(k+i-1|k)$ on the system (1), and $x(k|k)$ is the state of the system measured at time k.

We compute the optimal control according to the linear state-feedback law:

$$u(k+i|k) = F(k)x(k+i|k), \quad i \geq 0, \tag{5}$$

where

$F(k) \in \mathbb{R}^{n_u \times n_x}$ is the state-feedback matrix at time k.

We solve this problem by minimizing an upper bound on the objective function $J(k)$. We derive an upper bound on our objective function $J(k)$ and at each sampling time k we calculate predictive control $u(k+i|k) = F(k)x(k+i|k), i \geq 0$, so to minimize this upper bound. At time k only the first control move $u(k) = u(k|k)$ is implemented and we get the feedback control for the current state $x(k)$. Then the state $x(k+1)$ is measured and the optimization is repeated at the next sampling time $k+1$.

Main Results

The following theorem gives the state-feedback matrix.

Theorem *The state-feedback matrix of the control low (5) which minimizes the upper bound on $J(k)$ at sampling time k is given by:*

$$F(k) = Y(k)S(k)^{-1}, \tag{6}$$

where the matrices $S(k) = S(k)^T > 0$ and $Y(k)$ are the solutions to the following eigenvalue problem (EVP):

$$\min_{\gamma(k)>0, S(k)=S(k)^T>0, Y(k)} \gamma(k) \tag{7}$$

subject to

$$\begin{pmatrix} 1 & x(k)^T \\ x(k) & S(k) \end{pmatrix} \geq 0,$$

and

$$\begin{pmatrix} S(k) & C_{0l}^T & \dots & C_{nl}^T & S(k)Q^{1/2} & Y(k)^T R^{1/2} \\ C_{0l} & S(k) & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ C_{nl} & 0 & \dots & S(k) & 0 & 0 \\ Q^{1/2}S(k) & 0 & \dots & 0 & \gamma(k)I & 0 \\ R^{1/2}Y(k) & 0 & \dots & 0 & 0 & \gamma(k)I \end{pmatrix} \geq 0, \quad l = 1, \dots, L,$$

where

$C_{jl} = A_{jl}S(k) + B_{jl}Y(k),\ j = 0, \dots, n,$

I *is a unit matrix,* 0 *is a zero matrix of suitable dimensions,*

the signs "> 0", "≥ 0" *denote the matrices to be positive definite or positive semidefinite.*

As a result we get the optimal robust control strategy providing the system with stability in the mean-square sense:

$$E\left\{x(k+i|k)x(k+i|k)^T\middle|x(k)\right\} \to 0 \quad \text{for} \quad i \to \infty.$$

The problem (7) is concerned with the class of convex optimization problems with a liner goal function and linear matrix inequalities (LMI) constraints. There are effective numerical methods for solving such of problems.

References:

[1] M.V. Kothare, V. Balakrishnan, M. Morari, Robust constrained model predictive control using linear matrix inequality, *Automatica*, Vol. 32 (1996), No. 10, pp. 1361–1379.

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Numerical Example

Consider the system described by following equation:

$$x(k+1) = \Big(A_0(k) + A_1(k)w(k)\Big)x(k) + \Big(B_0(k) + B_1(k)w(k)\Big)u(k),$$

$$k = 0, 1, 2, \ldots,$$

where

$$A_0(k) = \begin{pmatrix} 1 & 0.1 \\ 0 & 1-\alpha(k) \end{pmatrix}, \quad A_1(k) = \begin{pmatrix} \beta(k) & 0 \\ 0 & 0.9 \end{pmatrix},$$

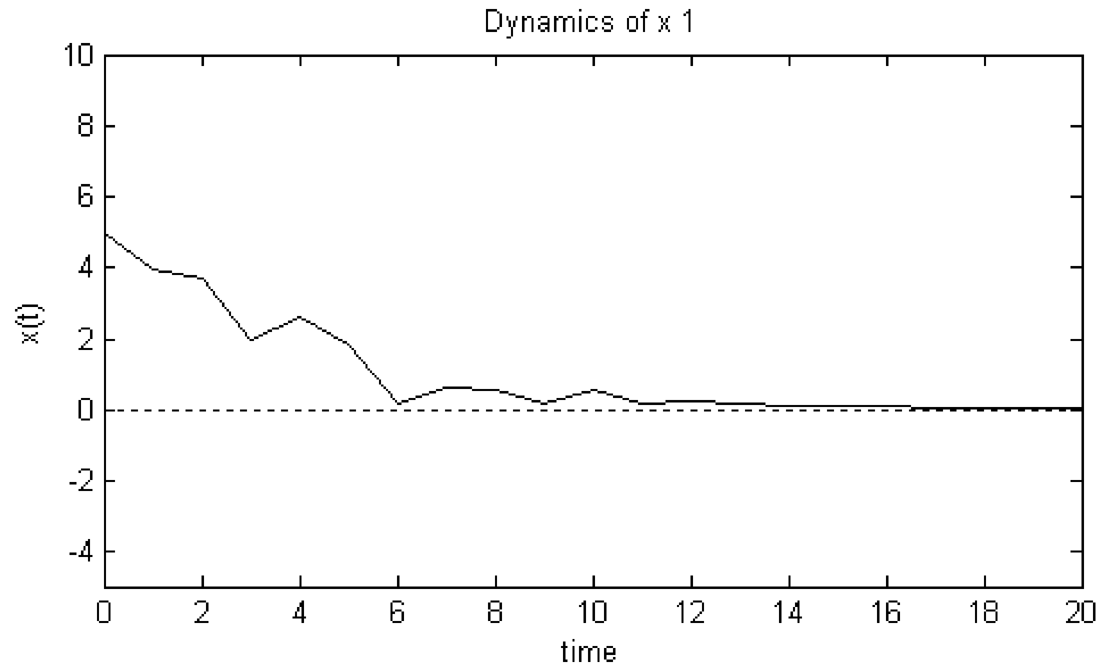
$$B_0(k) = \begin{pmatrix} 0.5\alpha(k) & 0 \\ 0 & 0.3 \end{pmatrix}, B_1(k) = \begin{pmatrix} \beta(k) & 0 \\ 0 & 0 \end{pmatrix},$$

$$\alpha(k) = [0.1, 0.7], \beta(k) = [0.2, 0.8].$$

The weighting matrices of the performance objective are

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}.$$

The next figures show the simulation results. The initial state $x(0) = [5 -5]^T$.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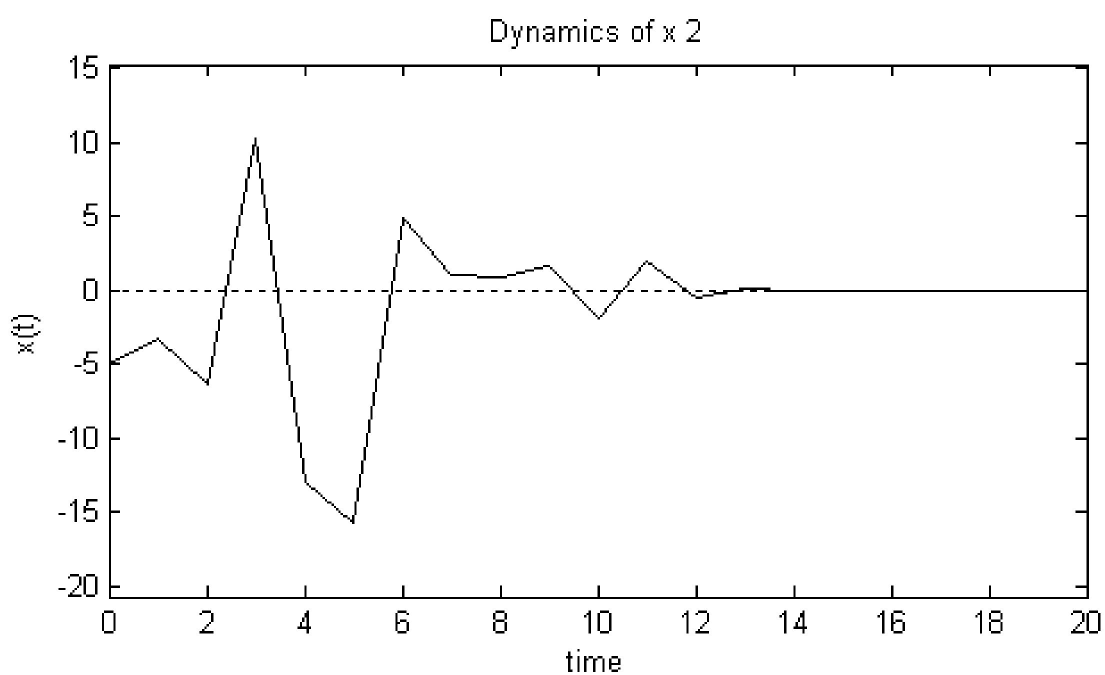
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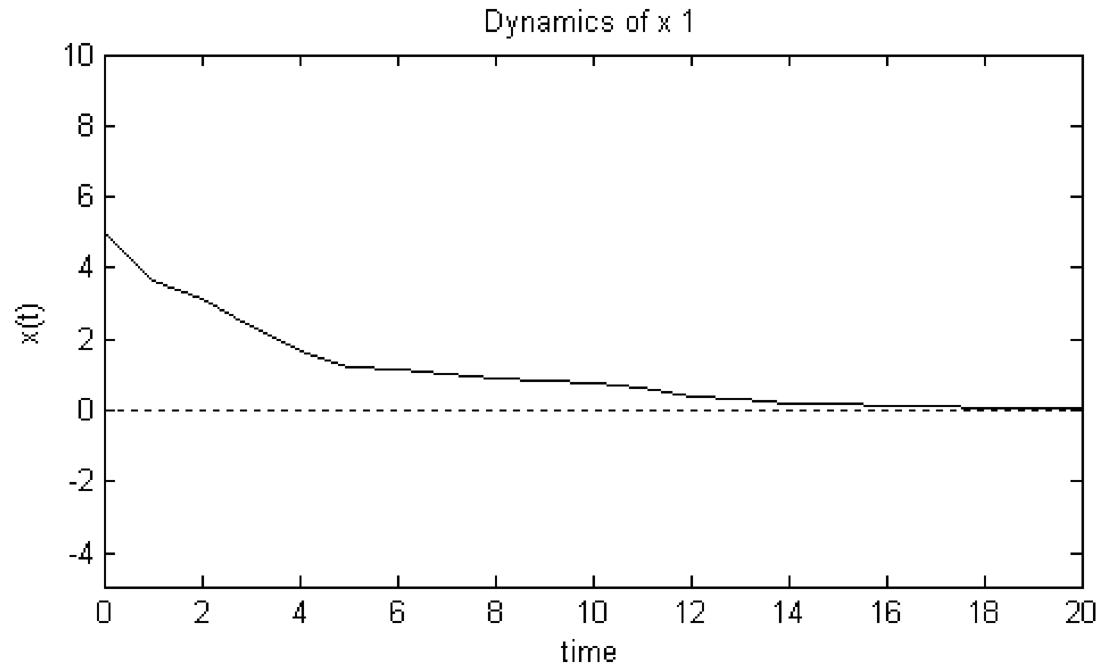
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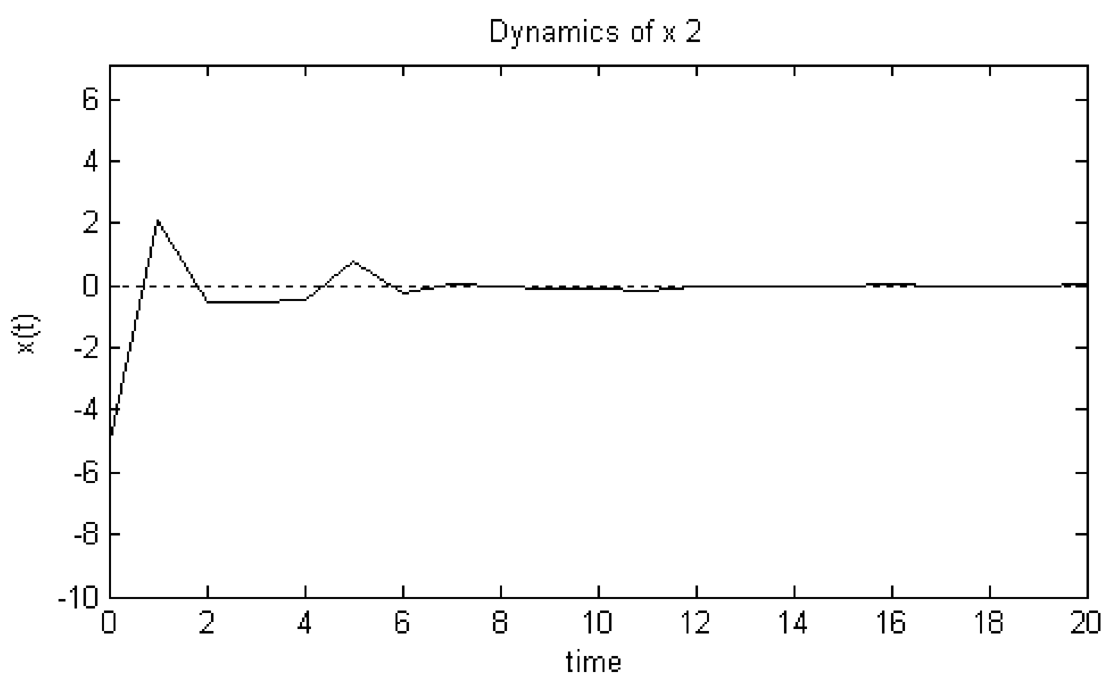
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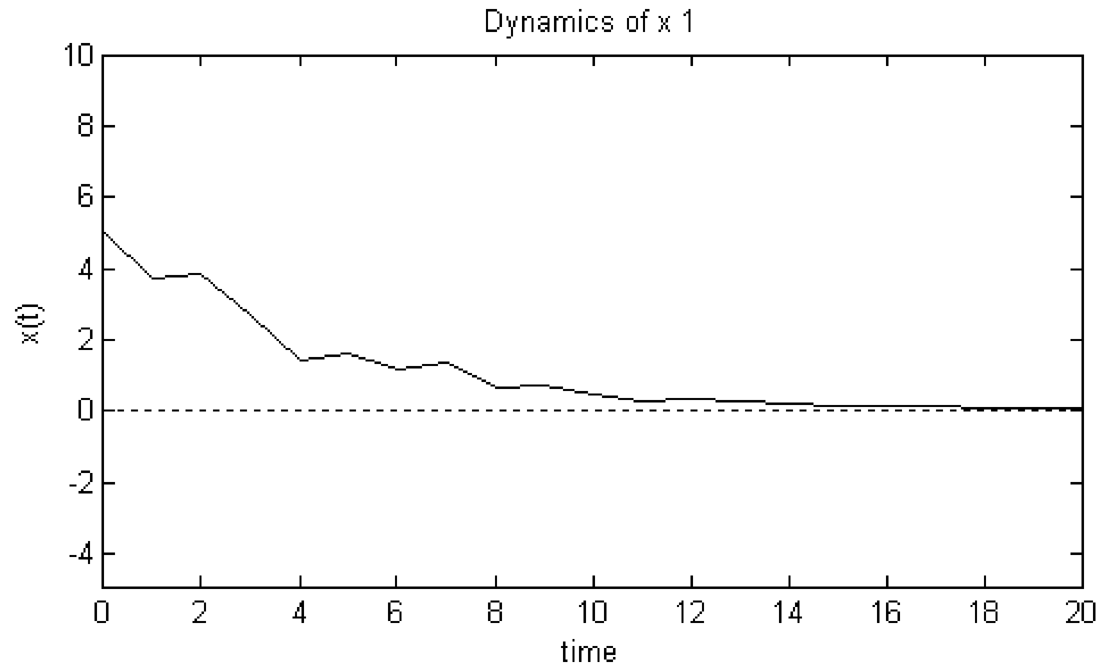
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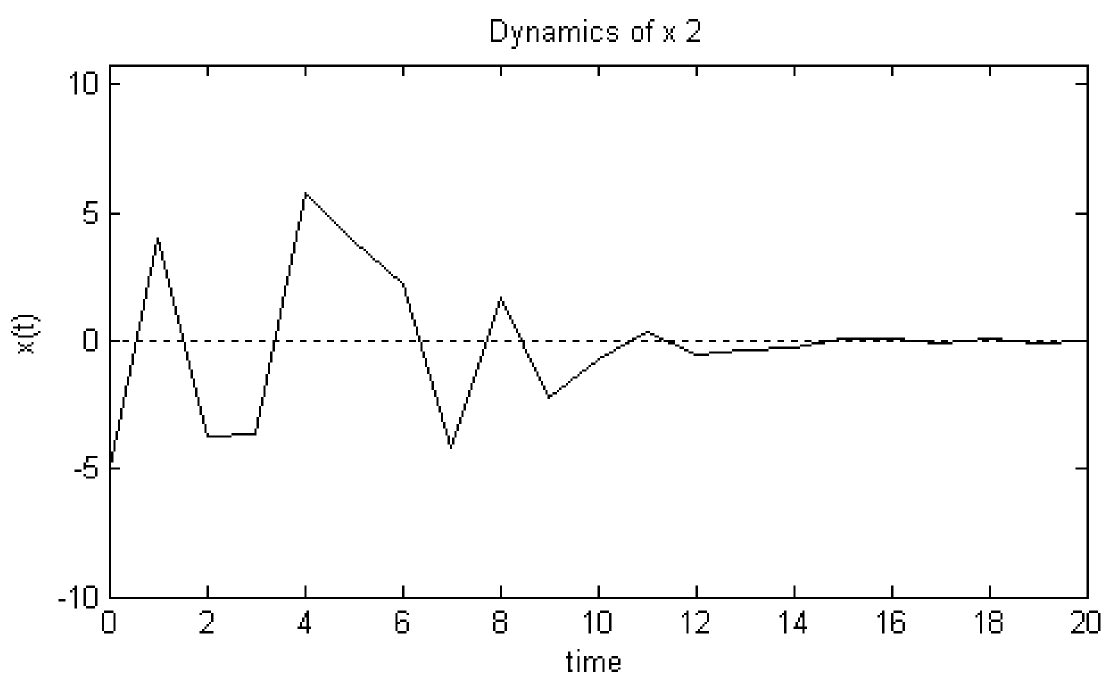
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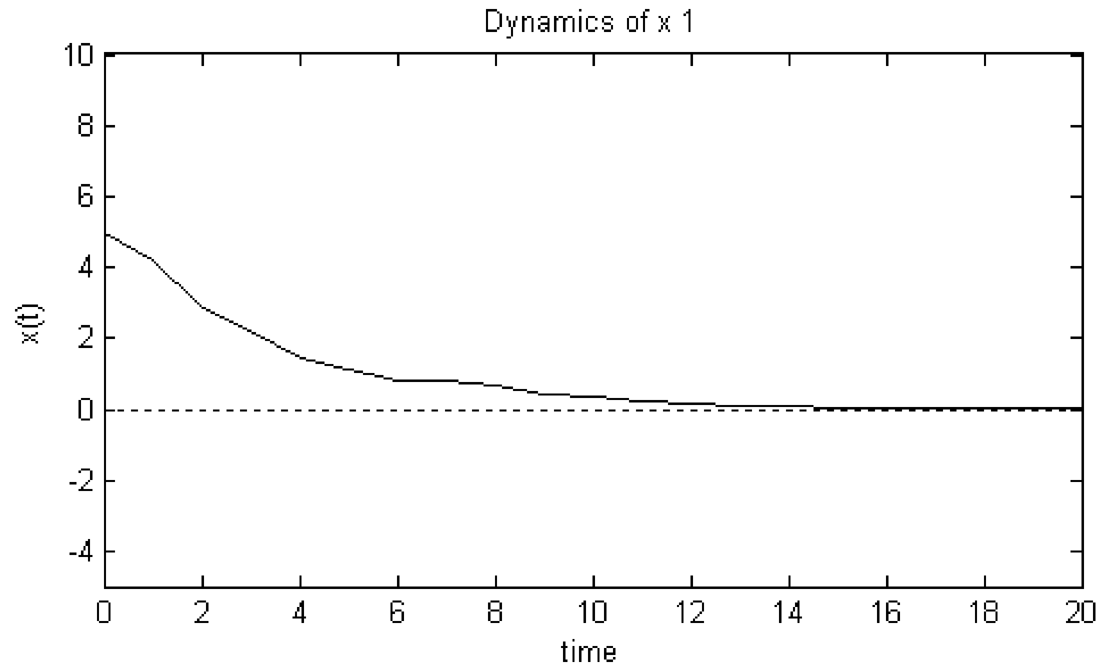
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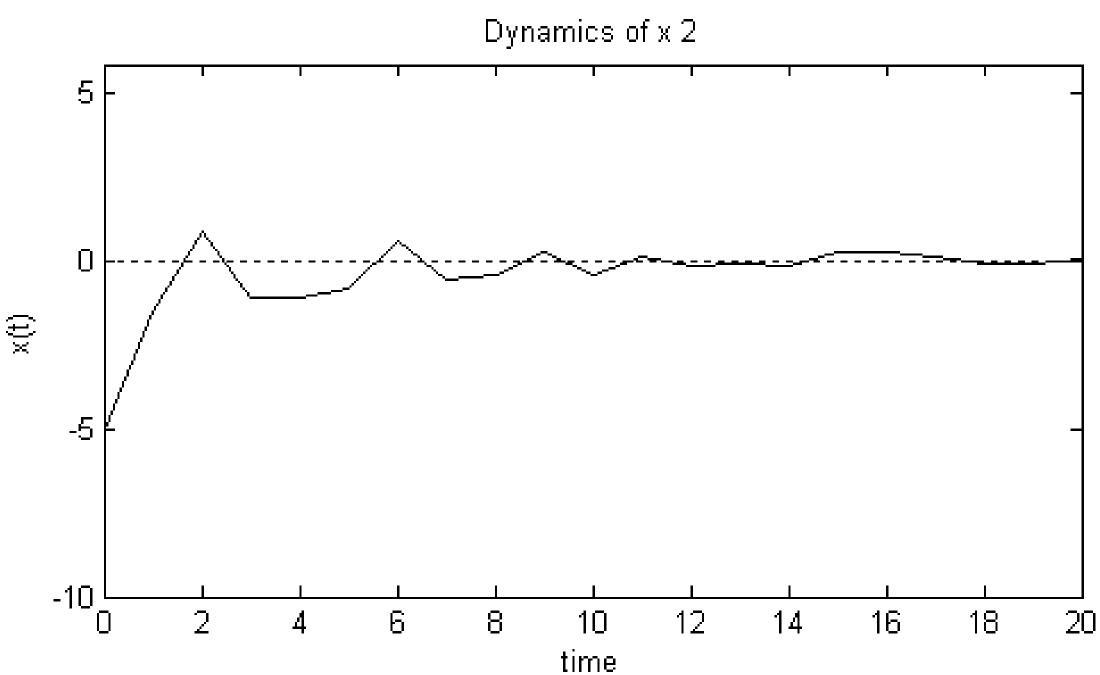
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