On boundedness and unboundedness of polyhedral estimates for reachable sets of linear systems

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Definition of parallelepiped

Parallelepiped in $\mathbb{R}^n$:

$$\mathcal{P} = \mathcal{P}(p, P, \pi) = \{x \mid x = p + \sum_{i=1}^{n} p^i \pi_i \xi_i,\ |\xi_i| \le 1\}.$$

$p \in \mathbb{R}^n$ — center of parallelepiped;

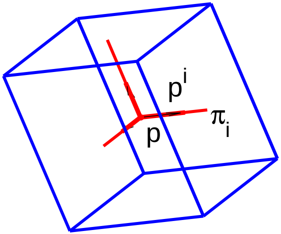
$P = \{p^i\} \in \mathcal{M}_*^{n\times n}$ — orientation matrix;

$\mathcal{M}_*^{n\times n} = \{P \in \mathbb{R}^{n\times n} \mid \det P \neq 0,\ \|p^i\| = 1\}$;

p^i — directions of "semi-axes";

$\pi \in \mathbb{R}^n,\ \pi_i \ge 0$ — values of "semi-axes".

($\|p^i\| = 1$ — is not important).



Parallelepiped $\mathcal{P}$ with $P = I$ is known as a box or an interval vector.

External polyhedral estimate $\mathcal{P}$ for $\mathcal{Q} \subset \mathbb{R}^n$:

$$\mathcal{Q} \subseteq \mathcal{P} = \mathcal{P}(p, P, \pi).$$

Tight [2] (in direction l) external estimate $\mathcal{P}$ for $\mathcal{Q}$:

$$\mathcal{Q} \subseteq \mathcal{P} \quad \text{and} \quad \exists\ l \in \mathbb{R}^n: \ \rho(\pm l|\mathcal{P}) = \rho(\pm l|\mathcal{Q}).$$

Touching external estimate $\mathcal{P}(p, P, \pi)$ for $\mathcal{Q}$:

it is tight estimate in directions $l^i = {P^{-1}}^\top \mathrm{e}^i, \quad i = 1, \ldots, n.$

($\rho(l|\mathcal{Q}) = \sup\{l^\top x \,|\, x \in \mathcal{Q}\}$ — support function, e^i — unit vector oriented along the axis $0x_i$).

Consider a linear system:

$$\dot{x} = A(t)\,x + w(t), \quad t \in \mathcal{T} = [0, \theta]. \tag{1}$$

$$x(0) \in \mathcal{X}_0; \quad w(t) \in \mathcal{R}(t). \tag{2}$$

Reachable set for system (1) – (2):

$\mathcal{X}(t) = \mathcal{X}(t, 0, \mathcal{X}_0) = \{\ x \in \mathbb{R}^n : \exists\ \{x(0), w(\cdot)\}$, that satisfies (2) and generates a solution $x(\cdot)$ of (1) satisfying $x(t) = x\ \}$.

It is known that reachable sets satisfy the semigroup property.

We suppose the sets $\mathcal{X}_0$, $\mathcal{R}(t)$ to be parallelepipeds:

$$\mathcal{X}_0 = \mathcal{P}(p_0, P_0, \pi_0), \quad \mathcal{R}(t) = \mathcal{P}(r(t), R(t), \rho(t)). \tag{3}$$

Problems

Problems considered earlier:

- Find some external estimates $\mathcal{P}(t)=\mathcal{P}(p(t),P(t),\pi(t))$ for $\mathcal{X}(t)$: $\mathcal{X}(t)\subseteq\mathcal{P}(t)$, satisfying evolutionary properties (the "upper" semigroup property and the superreachability property) for $\mathcal{P}(t)$
  which are analogues to the semigroup property for $\mathcal{X}(t)$.
  Moreover, describe a parametrized family $\mathfrak{P}$ of such estimates.
- Introduce some families of tight/touching estimates $\mathcal{P}(\cdot)$ such that

$$\mathcal{X}(t)=\bigcap\mathcal{P}(t).$$

Family $\mathfrak{P}$ of external estimates $\mathcal{P}(\cdot)$

Fix $P(\cdot) \in C^1$: $\det P(t) \neq 0, t \in \mathcal{T}$ (dynamics of orientation matrices). Let $p(\cdot)$ and $\pi(\cdot)$ satisfy

$$\dot{p} = A\,p + r, \quad p(0) = p_0;$$

$$\dot{\pi} = \mathrm{Ab}\,(P^{-1}(AP - \dot{P}))\pi + \mathrm{Abs}\,(P^{-1}R)\rho, \;\; \pi(0) = \mathrm{Abs}\,(P(0)^{-1}P_0)\pi_0,$$

where $(\mathrm{Abs}\,A)_i^j = |a_i^j|$ for $A = \{a_i^j\}$ and $(\mathrm{Ab}\,A)_i^i = a_i^i$, $(\mathrm{Ab}\,A)_i^j = |a_i^j|$, $i \neq j$.

Theorem 1

Parallelepipeds $\mathcal{P}(t) = \mathcal{P}(p(t), P(t), \pi(t))$ satisfy the generalized upper semigroup property and the superreachability property and $\mathcal{X}(t) \subseteq \mathcal{P}(t)$.

Here the entire family $\mathfrak{P}$ of estimates is described ($P(\cdot)$ serves as a parameter).

Subfamilies of estimates with different dynamics $P(\cdot)$

Subfamily $\mathfrak{P}^1 \subset \mathfrak{P}$ (of touching estimates $\mathcal{P}(\cdot)$):

$$P(\cdot) \text{ satisfies } \quad \dot{P} = AP,\ P(0) = V.$$

Proposition 1 (about estimates $\mathcal{P}(\cdot) \in \mathfrak{P}^1$):

$\mathcal{P}(t)$ are touching for $\mathcal{X}(t)$ and $\mathcal{X}(t) = \bigcap \{\mathcal{P}(t) \mid V \in \mathcal{V}^0\}$, $t \in \mathcal{T}$.

Subfamily $\mathfrak{P}^2 \subset \mathfrak{P}$ of estimates with constant orientation matrices (includes box-valued or coordinate-wise estimates):

$$P(t) \equiv P = V.$$

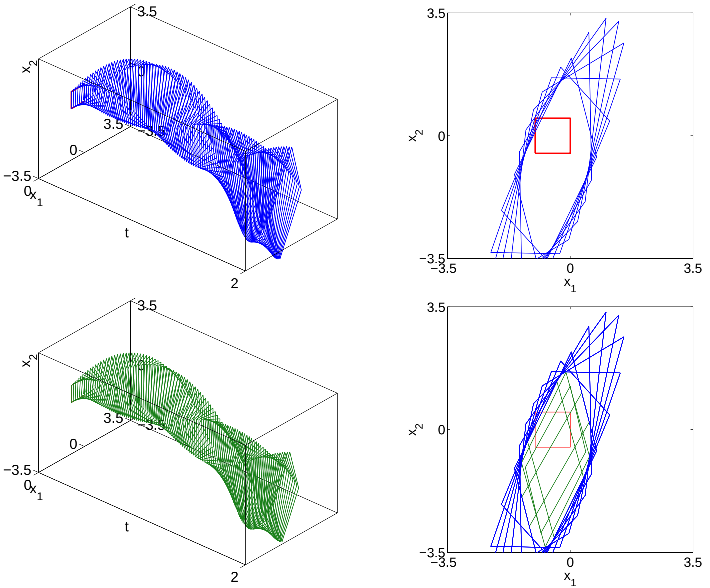
Subfamily $\mathfrak{P}^3 \subset \mathfrak{P}$ (of tight estimates $\mathcal{P}(\cdot)$):

$$\dot{p}^i = A(t)p^i,\ \ i{=}1,\dots,n{-}1;\ \ \dot{p}^n = -A(t)^\top p^n;$$
$$P(0) = V = \{v^i\}, \quad \det V \neq 0, \quad {v^n}^\top v^i = 0,\ \ i{=}1,\dots,n{-}1.$$

Proposition 2 (about estimates $\mathcal{P}(\cdot) \in \mathfrak{P}^3$):

$\mathcal{P}(t)$ are tight (in directions $p^n(t)$) estimates for $\mathcal{X}(t)$ and $\mathcal{X}(t) = \bigcap \{\mathcal{P}(t) \mid v^n \in \mathbb{R}^n, \|v^n\| = 1\}$, $t \in \mathcal{T}$.

Example of external and internal estimates



Problems

Assumption 1

In system (1) matrix $A(t) \equiv A$ is stable (i.e. all $\mathrm{Re}\,\lambda_k < 0$) and set-valued map $\mathcal{R}(t)$ ($t \in [0, \infty)$) is bounded.

Investigate boundedness and unboundedness of external estimates from $\mathfrak{P}^i$, $i = 1, 2, 3$:

- For which $P(0) = V$ estimates $\mathcal{P}(t)$, $t \in [0, \infty)$, are either bounded or unbounded?
- Find conditions on A, $\mathcal{P}_0$, $\mathcal{R}(\cdot)$ which ensure that
  - there exist bounded or unbounded estimates in $\mathfrak{P}^i$;
  - all the estimates from $\mathfrak{P}^i$ are bounded or unbounded.
- What is possible degree of increasing the estimates from $\mathfrak{P}^i$?

Comparison of estimates using a functional

Requirements on criterion $\mu(\mathcal{P}) = \mu(\mathcal{P}(p, P, \pi))$:

- it is defined $\forall \mathcal{P}$; non-negativity: $\mu(\mathcal{P}) \geq 0$.
- monotonicity under inclusion: $\mathcal{P}^{(1)} \subseteq \mathcal{P}^{(2)} \Rightarrow \mu(\mathcal{P}^{(1)}) \leq \mu(\mathcal{P}^{(2)})$.

Volume functional: $\mu_{\mathrm{vol}}(\mathcal{P}) \stackrel{\triangle}{=} 2^{-n} \mathrm{vol}\, \mathcal{P} = |\det P| \prod_{i=1}^{n} \pi_i$.

Other possible criteria:

$\mu(\mathcal{P}) = \|q\|$, where $q = (\mathrm{Abs}\, P)\, \pi$ (we have $q_i = \rho(\pm \mathrm{e}^i | \mathcal{P} - p)$),
$\|q\|$ is arbitrary of usual norms $\|q\|_1$, $\|q\|_2$ or $\|q\|_\infty$.

Proposition 3

Boundedness (unboundedness) of $\mathcal{P}(\cdot)$ is equivalent to boundedness (unboundedness) of $\mu(\mathcal{P}(\cdot))$, where $\mu(\mathcal{P}) = \|q\|$.

Exponent $\chi = \chi(\mathcal{P})$ of the tube (estimate) $\mathcal{P}(t)$, $t \in [0, \infty)$:

$$\chi = \chi(\mathcal{P}) = \overline{\lim}_{t \to \infty} t^{-1} \ln \mu(\mathcal{P}(t))$$

Boundedness (unboundedness) of external estimates

Sufficient conditions for $\mathcal{P}(\cdot) \in \mathfrak{P}^i$, $i = 1, 2, 3$, to be bounded / unbounded (particularly depending on V, A, $\mathcal{P}_0$, $\mathcal{R}(\cdot)$) are obtained.

▸ $\dot{P} = AP$ ▸ $P(t) \equiv P$ ▸ $\mathfrak{P}^3$

We will see that the estimates can be unbounded not only if $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ (for $V = I$ this is "wrapping effect" known from interval analysis) but also if $\mathcal{P}(\cdot) \in \mathfrak{P}^1$ or $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ under the following

Condition of nondegeneracy of $\mathcal{R}(\cdot)$:

$$\mathcal{R}(t) \supseteq \mathcal{P}(r(t), I, \varepsilon_0 \,\mathrm{e}), \quad t \in [0, \infty), \quad \text{where } \varepsilon_0 > 0,\ \mathrm{e}=(1, \ldots, 1)^\top.$$

Estimates from $\mathfrak{P}^2$ can be unbounded also under the following

Condition of nondegeneracy of $\mathcal{P}_0$:

$$\mathcal{P}_0 \supseteq \mathcal{P}(p_0, I, \varepsilon_0 \,\mathrm{e}), \quad t \in [0, \infty), \quad \text{where } \varepsilon_0 > 0.$$

▸ $\dot{P} = AP$ ▸ $P(t) \equiv P$

Auxiliary material: real Jordan form of matrix

If $A \in \mathbb{R}^{n\times n}$ and $\lambda_k = \alpha_k + \beta_k\sqrt{-1}$, $k = 1, \dots, m$, are all its eigenvalues (here $\beta_k \geq 0$), then A is similar to

Matrix in the real Jordan form

$$J = TAT^{-1}, \quad \text{where} \quad J = \operatorname{diag}\{J_1, \dots, J_m\};$$

$$J_k = \begin{bmatrix} S_k & I & \dots & 0 & 0 \\ 0 & S_k & \dots & 0 & 0 \\ & & \dots & & \\ 0 & 0 & \dots & S_k & I \\ 0 & 0 & \dots & 0 & S_k \end{bmatrix} \in \mathbb{R}^{(\nu_k\gamma_k)\times(\nu_k\gamma_k)}, \;\; k = 1, \dots, m;$$

$$S_k, I, 0 \in \mathbb{R}^{\nu_k\times\nu_k}, \quad \nu_k = 1 \text{ or } 2;$$

$$\nu_k = 1,\; S_k = \alpha_k \text{ if } \beta_k = 0;\; \nu_k = 2,\; S_k = \begin{bmatrix} \alpha_k & -\beta_k \\ \beta_k & \alpha_k \end{bmatrix} \text{ if } \beta_k \neq 0.$$

Matrix A is called diagonalizable if all $\gamma_k = 1$ and defective otherwise.

Theorem for $\mathfrak{P}^1$

Boundedness (unboundedness) of $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ (when $P(t) \equiv P$)

Let λ_k and ω_k be eigenvalues of matrices A and $A_P = \mathrm{Ab}\,(P^{-1}AP)$ respectively (recall that $(\mathrm{Ab}\,B)^i_i = b^i_i$, $(\mathrm{Ab}\,B)^j_i = |b^j_i|$, $i \neq j$).

Proposition 4

1. $\mathcal{P}(\cdot) \in \mathfrak{P}^2$ is bounded if $A_P = \mathrm{Ab}\,(P^{-1}AP)$ is stable.
2. $\mathcal{P}(\cdot)$ is unbounded if $\exists \omega_k$ with $\mathrm{Re}\,\omega_k > 0$ and either $\mathcal{P}_0$ or $\mathcal{R}(\cdot)$ satisfy Condition of nondegeneracy.

Theorem 2

1. If $A = \alpha I$, then $\mathcal{P}(\cdot)$ are bounded $\forall P$.
2. If $A \neq \alpha I$ and either $\mathcal{P}_0$ or $\mathcal{R}(\cdot)$ satisfy Condition of nondegeneracy, then $\exists \mathcal{P}(\cdot) \in \mathfrak{P}^2$ with arbitrary large $\chi(\mathcal{P})$.
3. If all $|\mathrm{Im}\,\lambda_k| < |\mathrm{Re}\,\lambda_k|$, then $\exists\, P$ (in particular, $P = T^{-1}$, where $A = T^{-1}JT$) which generate bounded $\mathcal{P}(\cdot)$.

Boundedness (unboundedness) of $\mathcal{P}(\cdot) \in \mathfrak{P}^1$ (touching estimates) when $\dot{P} = AP$

Proposition 5

If $\mathcal{R}(t)$ (which bound the controls) are singletons, then estimates

$$\mathcal{P}(t) = \mathcal{P}(p(t), P(t), \pi(t)) \to p(t) \quad \text{as } t \to \infty, \quad \forall\, P(0).$$

Let $\mathrm{m} = \min |\mathrm{Re}\, \lambda_k|$, $\mathrm{M} = \max |\mathrm{Re}\, \lambda_k|$.

Theorem 3

1. $\chi(\mathcal{P}) \le \mathrm{M} - \mathrm{m}$.
2. If A is diagonalizable, then there are following possibilities.
    1. If $\mathrm{M} = \mathrm{m}$, then $\mathcal{P}(\cdot)$ are bounded $\forall\, P(0)$.
    2. If $\mathrm{M} \neq \mathrm{m}$, then $\exists\, P(0)$ (in particular, $P(0) = T^{-1}$ ▸) which generate bounded estimates $\mathcal{P}(\cdot)$.
    But if $\mathcal{R}(\cdot)$ satisfies Condition of nondegeneracy ▸, then $\exists\, P(0)$ which generate unbounded estimates $\mathcal{P}(\cdot)$.
3. If A is defective and $\mathcal{R}(\cdot)$ satisfies Condition of nondegeneracy, then $\mathcal{P}(\cdot)$ are unbounded $\forall\, P(0)$.

Recall that $P(0) = V = \{v^i\}$ satisfies ${v^n}^\top v^i = 0$, $i = 1, \ldots, n{-}1$.
Let $\bar{V} = \{\bar{v}^i\}$ be such that $\bar{v}^i = v^i$, $i = 1, \ldots, n{-}1$, $\det \bar{V} \neq 0$.

Theorem 4

1. $\chi(\mathcal{P}) \leq \mathrm{M} - \mathrm{m}$, where $\mathrm{m} = \min |\mathrm{Re}\, \lambda_k|$, $\mathrm{M} = \max |\mathrm{Re}\, \lambda_k|$.
2. If A is diagonalizable, then there are following possibilities.
   1. If $\mathrm{M} = \mathrm{m}$, then $\mathcal{P}(\cdot)$ are bounded $\forall\, P(0)$.
   2. If $\mathrm{M} \neq \mathrm{m}$, then $\exists\, P(0)$ which generate bounded estimates $\mathcal{P}(\cdot)$ (in particular, $P(0) = V$ for which the corresponding matrix $\bar{V} = T^{-1}$, where T is such that $J = TAT^{-1}$). But if $n \geq 3$ and $\mathcal{R}(\cdot)$ satisfies Condition of nondegeneracy, then $\exists\, P(0)$ which generate unbounded estimates $\mathcal{P}(\cdot)$.

The fact which is unlike to the situation for $\mathfrak{P}^1$ and $\mathfrak{P}^2$:

Theorem 5

If $n = 2$, then $\mathcal{P}(\cdot)$ are bounded $\forall\, P(0)$.

Boundedness (unboundedness) of $\mathcal{P}(\cdot) \in \mathfrak{P}^i$, $i = 1, 2, 3$ (case $n = 2$)

Additional conditions on $\mathcal{P}_0, \mathcal{R}(\cdot)$	$\mathrm{Im}\,\lambda_1 = \mathrm{Im}\,\lambda_2 = 0$						$\lambda_{1,2} = \alpha \pm \beta\sqrt{-1}$, $\beta \neq 0$					
	A—diagonalizable: $J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$				A—defective: $J = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix}$		A — diagonalizable: $J = TAT^{-1} = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$					
	$\lambda_1 = \lambda_2$		$\lvert\lambda_1\rvert < \lvert\lambda_2\rvert$		$\lambda_1 = \lambda_2 = \alpha$		$\lvert\alpha\rvert > \lvert\beta\rvert$		$\lvert\alpha\rvert = \lvert\beta\rvert$		$\lvert\alpha\rvert < \lvert\beta\rvert$	
	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$	$\mathfrak{P}^1$	$\mathfrak{P}^2$
–	green	green	yellow	yellow		yellow	green	yellow	green		green	
$\mathcal{R}(t) \equiv r(t)$, $\mathrm{int}\,\mathcal{P}_0 \neq \emptyset$	green	green	green	yellow, brown	green	yellow, brown	green	yellow, brown	green	yellow, brown	green	red
$\mathrm{int}\,\mathcal{R}(t) \neq \emptyset$	green	green	yellow, brown	yellow, brown	red	yellow, brown	green	yellow, brown	green	red	green	red

- green — all tubes are bounded;
- yellow — $\exists$ bounded tubes;
- brown — $\exists$ unbounded tubes;
- red — all tubes are unbounded.

If $n = 2$, then all $\mathcal{P}(\cdot) \in \mathfrak{P}^3$ are bounded. $\Rightarrow$ green should be everywhere for $\mathfrak{P}^3$.

Family $\mathfrak{P}^0 \not\subseteq \mathfrak{P}^+$ of estimates for time-invariant systems ($P(t) \equiv P$)

Assumption 2

$A(t) \equiv A$ and $\mathcal{R}(t) \equiv \mathcal{R}$, $\rho(t) \equiv \rho$.

Family $\mathfrak{P}^0 \not\subseteq \mathfrak{P}^+$ of $\mathcal{P}(p(\cdot), P, \pi(\cdot))$ with $P(t) \equiv P$:

$$\pi(t) = \pi^1(t) + \pi^2(t);$$
$$\pi^1(t) = (\mathrm{Abs}\, P^1(t))\,\pi_0; \qquad \dot{P}^1 = \tilde{A}\,P^1, \quad P^1(0) = P^{-1}P_0;$$
$$\dot{\pi}^2 = (\mathrm{Abs}\, P^2(t))\,\rho, \quad \pi^2(0) = 0; \qquad \dot{P}^2 = \tilde{A}\,P^2, \quad P^2(0) = P^{-1}R;$$
$$\tilde{A} = P^{-1}AP.$$

Proposition 6

Under Assumption 2, if $\mathcal{P}(\cdot) \in \mathfrak{P}^0$, then $\mathcal{P}(t)$ are touching estimates for $\mathcal{X}(t)$ and $\mathcal{X}(t) = \bigcap \{\mathcal{P}(t) \mid P \in \mathcal{V}^0\}$.
Under additional Assumption 1 $\mathcal{P}(\cdot)$ is bounded $\forall\, P \in \mathcal{M}_*^{n\times n}$.

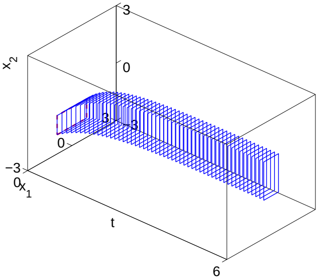
Such estimates $\mathcal{P}(t)$ *do not* satisfy the evolutionary properties.

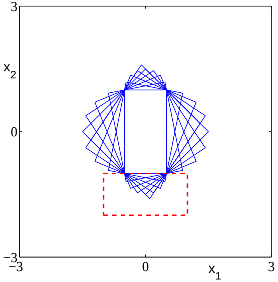
Example 1: $\operatorname{Im}\lambda_k = 0$, $\lambda_1 = \lambda_2$, A — diagonalizable

$$A \equiv \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \begin{array}{l} \mathcal{X}_0 = \mathcal{P}((0,-1.5)^\top, I, (1,0.5)^\top), \\ \mathcal{R} = \mathcal{P}(0, I, (0.5,1)^\top), \qquad \theta = 6. \end{array}$$

$(\lambda_1 = \lambda_2 = -1)$

Obtained estimates from $\mathfrak{P}^1$, $\mathfrak{P}^2$, $\mathfrak{P}^3$ and $\mathfrak{P}^0$ coincide:



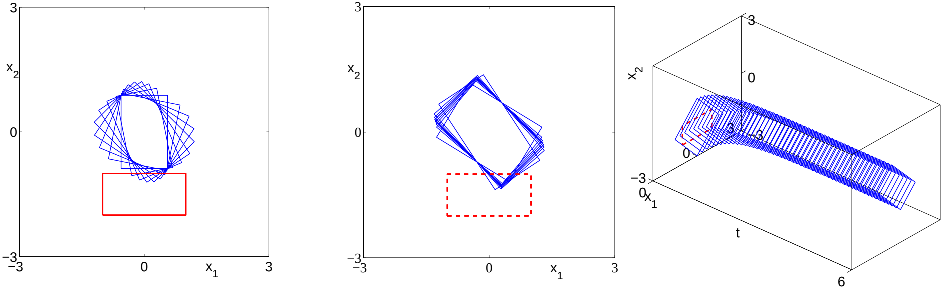


Example 2: $\operatorname{Im}\lambda_k = 0,\ \lambda_1 \neq \lambda_2,\ A$ — diagonalizable

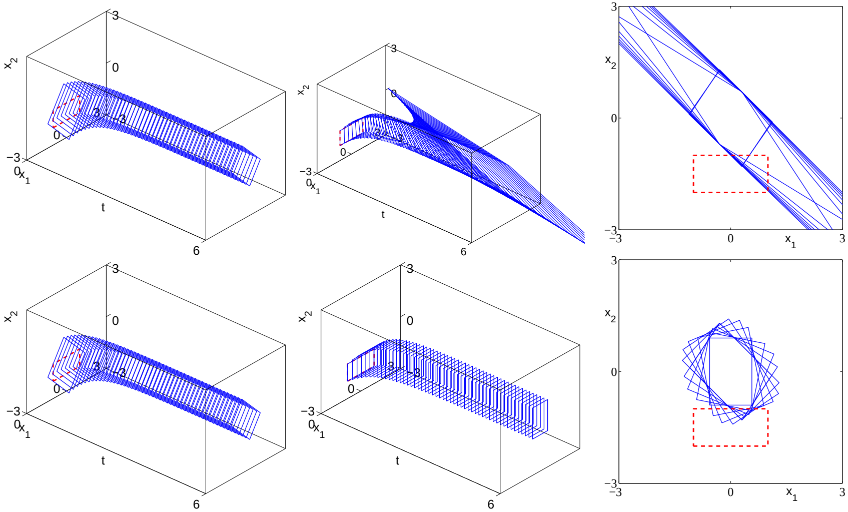
$$A \equiv \begin{bmatrix} -1.2 & -0.2 \\ -0.3 & -1.3 \end{bmatrix}, \quad \begin{array}{l} \mathcal{X}_0 = \mathcal{P}((0,-1.5)^\top, I, (1, 0.5)^\top), \\ \mathcal{R} = \mathcal{P}(0, I, (0.5, 1)^\top), \qquad \theta = 6. \end{array}$$

$(\lambda_1 = -1,\ \lambda_2 = -1.5)$

Estimates from $\mathfrak{P}^0$ (at the left) and from $\mathfrak{P}^3$ (at the right):



Estimates from $\mathfrak{P}^1$ (at the top): ($\exists$ bounded, $\exists$ unbounded)
and from $\mathfrak{P}^2$ (at the bottom): ($\exists$ bounded, $\exists$ unbounded)

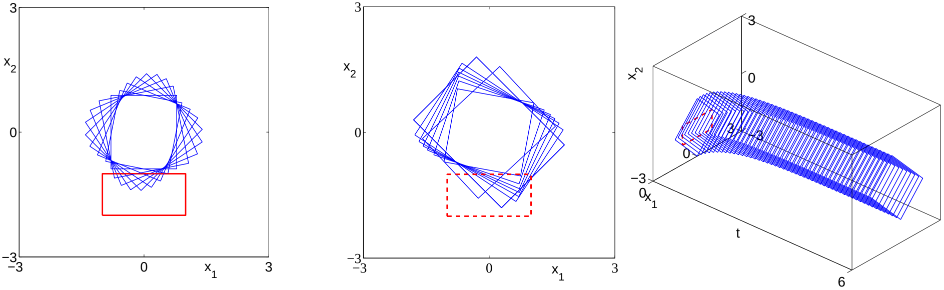


Example 3: $\mathrm{Im}\,\lambda_k = 0,\ \ \lambda_1 = \lambda_2,\ \ A$ — defective

$$A \equiv \begin{bmatrix} -0.8 & 0.2 \\ -0.2 & -1.2 \end{bmatrix}, \quad \begin{array}{l} \mathcal{X}_0 = \mathcal{P}((0,-1.5)^\top, I, (1,0.5)^\top), \\ \mathcal{R} = \mathcal{P}(0, I, (0.5,1)^\top), \qquad \theta = 6. \end{array}$$

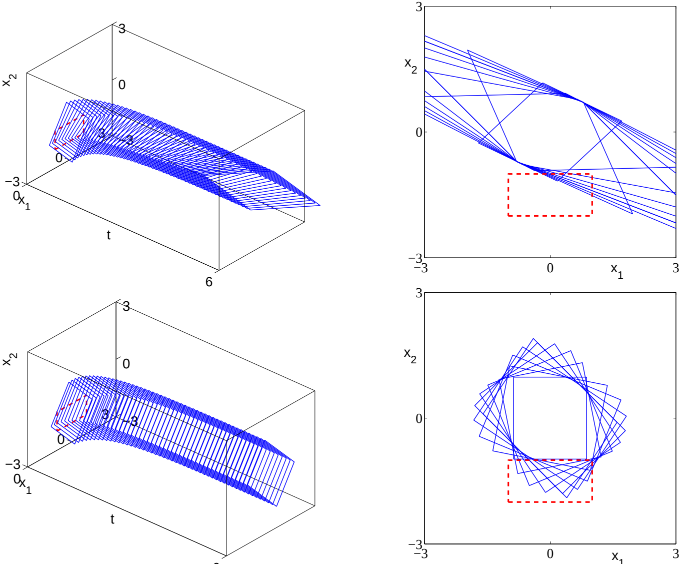
$(\lambda_1 = \lambda_2 = -1)$

Estimates from $\mathfrak{P}^0$ (at the left) and from $\mathfrak{P}^3$ (at the right):



Example 3: $\mathrm{Im}\,\lambda_k = 0$, $\lambda_1 = \lambda_2$, A — defective

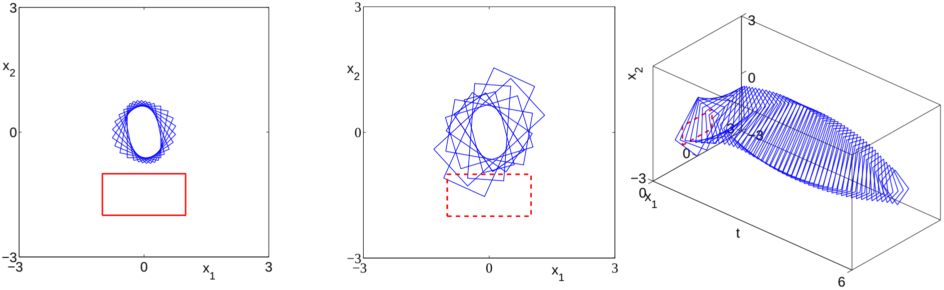
Estimates from $\mathfrak{P}^1$ (at the top): (all unbounded)
and from $\mathfrak{P}^2$ (at the bottom): ($\exists$ bounded, $\exists$ unbounded)



$$A \equiv \begin{bmatrix} -0.5 & -0.5 \\ 1 & -1.5 \end{bmatrix}, \quad \begin{array}{l} \mathcal{X}_0 = \mathcal{P}((0,-1.5)^\top, I, (1,0.5)^\top), \\ \mathcal{R} = \mathcal{P}(0, I, (0,1)^\top), \qquad \theta = 6. \end{array}$$

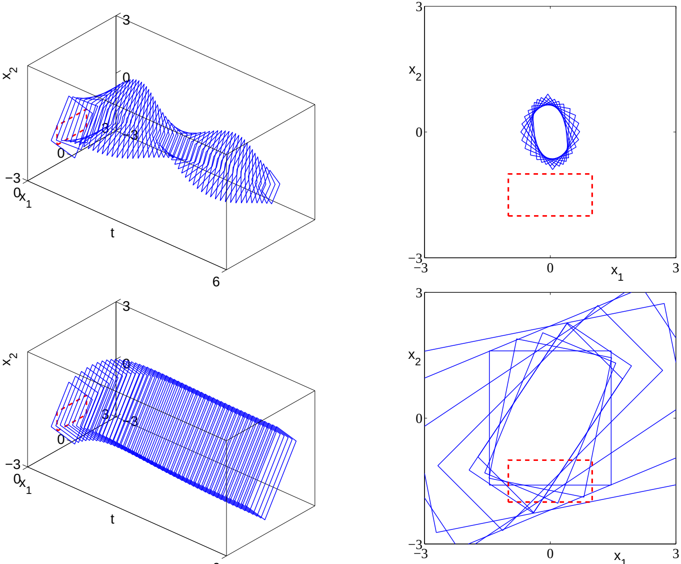
$(\alpha = -1, \; \beta = 0.5)$

Estimates from $\mathfrak{P}^0$ (at the left) and from $\mathfrak{P}^3$ (at the right):



Example 4: $\lambda_{1,2} = \alpha \pm \beta\sqrt{-1}, \ \ |\beta| < |\alpha|$

Estimates from $\mathfrak{P}^1$ (at the top): (all bounded)
and from $\mathfrak{P}^2$ (at the bottom): ($\exists$ bounded, $\exists$ unbounded)

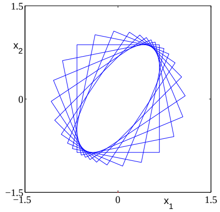


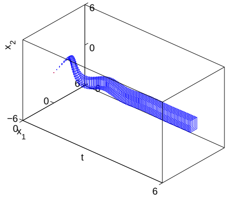
Example 5: $\lambda_{1,2} = \alpha \pm \beta\sqrt{-1}, \ \ |\beta| > |\alpha|$

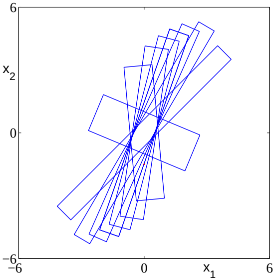
$$A \equiv \begin{bmatrix} 2.5 & -3.5 \\ 7 & -4.5 \end{bmatrix}, \quad \begin{array}{l} \mathcal{X}_0 = \mathcal{P}((0,-1.5)^\top, I, 0), \\ \mathcal{R} = \mathcal{P}(0, I, (0,1)^\top), \end{array} \qquad \theta = 6.$$

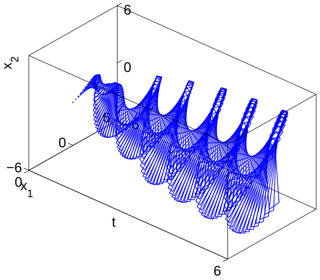
$(\alpha = -1,\ \beta = 3.5)$

Estimates from $\mathfrak{P}^0$ (at the left) and from $\mathfrak{P}^3$ (at the right):



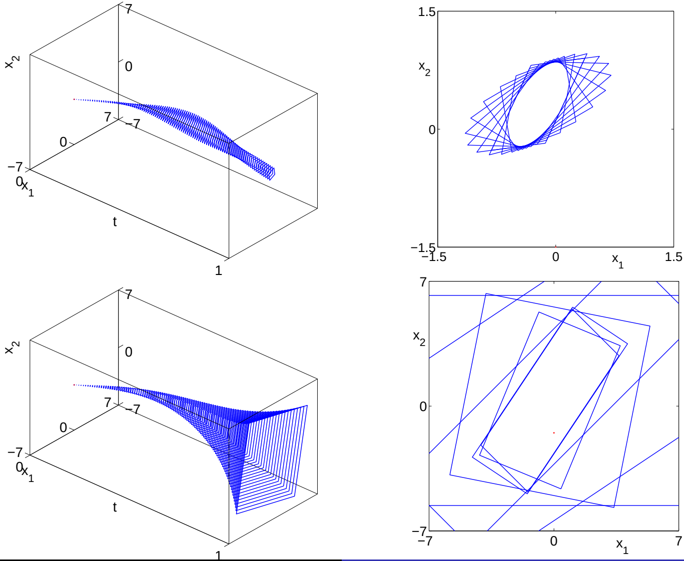






Example 5: $\lambda_{1,2} = \alpha \pm \beta\sqrt{-1}, \ \ |\beta| > |\alpha|$

$\theta = 1$. Estimates from $\mathfrak{P}^1$ (at the top): (all bounded) and from $\mathfrak{P}^2$ (at the bottom): (all unbounded).



Conclusion

Boundedness and unboundedness of estimates from $\mathfrak{P}^i \in \mathfrak{P}$, $i = 1, 2, 3$, is investigated for systems with stable constant matrices:

- Sufficient conditions for $\mathcal{P}(t)$, $t \in [0, \infty)$, to be bounded (unbounded), depending on V, A, $\mathcal{P}_0$, $\mathcal{R}(\cdot)$, are obtained.
- The conditions on A, $\mathcal{P}_0$, $\mathcal{R}(\cdot)$ are presented which ensure that either there exist bounded or unbounded estimates in $\mathfrak{P}^i$ or all the estimates from $\mathfrak{P}^i$ are bounded or unbounded.
- The possible degree of increasing the estimates from $\mathfrak{P}^i$ is described in terms of tube exponents.
- The full description, classification and comparison of possible situations of boundedness and unboundedness of estimates are given for two-dimensional systems.
- The results of numerical simulations are presented.

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Semigroup property

Semigroup property for reachable sets: ▸ Return 1

$$\mathcal{X}(t,0,\mathcal{X}_0) = \mathcal{X}(t,\tau,\mathcal{X}(\tau,0,\mathcal{X}_0)), \quad 0 \le \tau \le t \le \theta.$$

"Upper" semigroup property for $\mathcal{P}(t) = \boldsymbol{\mathcal{P}}(t,0,\mathcal{P}(0))$:

$$\boldsymbol{\mathcal{P}}(t,0,\mathcal{P}(0)) = \boldsymbol{\mathcal{P}}(t,\tau,\boldsymbol{\mathcal{P}}(\tau,0,\mathcal{P}(0))), \quad \forall \tau, t: \; 0 \le \tau \le t \le \theta; \quad \mathcal{X}_0 \subseteq \mathcal{P}(0).$$

Superreachability property for $\mathcal{P}(t)$: ▸ Return 2

$$\mathcal{X}(t,\tau,\mathcal{P}(\tau)) \subseteq \mathcal{P}(t), \quad \forall \tau, t: \; 0 \le \tau \le t \le \theta; \quad \mathcal{X}_0 \subseteq \mathcal{P}(0).$$

Proposition

If $\mathcal{R}(t)$ are singletons, then $\mathcal{P}(t) \to p(t) \quad$ as $t \to \infty$, $\forall\, P(0)$.

Theorem ▸ Nondeg. cond.

1. If A is diagonalizable, $\mathrm{M} = \mathrm{m}$, then $\mathcal{P}(\cdot)$ are bounded $\forall\, P(0)$.
2. Let A be diagonalizable, $\mathrm{M} \neq \mathrm{m}$, T be a matrix which reduces A to real Jordan form and $\tilde{V} = TV$, $\tilde{W} = V^{-1}T^{-1}$ be divided into blocks $\tilde{V}_i^j \in \mathbb{R}^{\nu_i \times \nu_j}$, $\tilde{W}_j^i \in \mathbb{R}^{\nu_j \times \nu_i}$ $(i, j = 1, \ldots, m)$.
   If V is such that for each pair λ_i, λ_j with $|\mathrm{Re}\lambda_i| < |\mathrm{Re}\lambda_j|$ we have $Z_i^j = 0 \in \mathbb{R}^{\nu_i \times \nu_j}$, where $Z_i^j = \sum_{k=1}^m \mathrm{Abs}\, \tilde{V}_i^k \, \mathrm{Abs}\, \tilde{W}_k^j$, then $\mathcal{P}(\cdot)$ is bounded.
   If V is such that $Z_i^j \neq 0 \in \mathbb{R}^{\nu_i \times \nu_j}$ for some pair λ_i, λ_j with $|\mathrm{Re}\lambda_i| < |\mathrm{Re}\lambda_j|$ and $\mathcal{R}(\cdot)$ satisfies Condition of nondegeneracy, then $\mathcal{P}(\cdot)$ is unbounded and $\chi(\mathcal{P}) \geq |\mathrm{Re}\lambda_j| - |\mathrm{Re}\lambda_i|$.
3. If A is defective and $\mathcal{R}(\cdot)$ satisfies Condition of nondegeneracy, then $\mathcal{P}(\cdot)$ are unbounded $\forall\, P(0)$.

Recall that $P(0) = V = \{v^i\}$ satisfies ${v^n}^\top v^i = 0$, $i = 1, \dots, n{-}1$.
Let $\bar{V} = \{\bar{v}^i\}$ be such that $\bar{v}^i = v^i$, $i = 1, \dots, n{-}1$, $\det \bar{V} \neq 0$.

Theorem ▸ Nondeg. cond.

1. If A is diagonalizable, $\mathrm{M} = \mathrm{m}$, then $\mathcal{P}(\cdot)$ are bounded $\forall\, P(0)$.
2. Let A be diagonalizable, $\mathrm{M} \neq \mathrm{m}$, T be a matrix which reduces A to real Jordan form and $\tilde{V} = T\bar{V}$, $\tilde{W} = \bar{V}^{-1}T^{-1}$ be divided into blocks $\tilde{V}_i^j \in \mathbb{R}^{\nu_i \times \nu_j}$, $\tilde{W}_j^i \in \mathbb{R}^{\nu_j \times \nu_i}$ $(i, j = 1, \dots, m)$. If V is such that for each pair λ_i, λ_j with $|\mathrm{Re}\lambda_i| < |\mathrm{Re}\lambda_j|$ we have $Z_i^j = 0 \in \mathbb{R}^{\nu_i \times \nu_j}$, where $Z_i^j = \sum_{k=1}^m \mathrm{Abs}\, \tilde{V}_i^k\, \mathrm{Abs}\, \tilde{W}_k^j$, then $\mathcal{P}(\cdot)$ is bounded.