Uses of Verified Methods for Solving Non-Smooth Initial Value Problems

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Non-smooth Models: Phenomena and Areas

	Friction	Impacts	Hysteresis
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Electrics	picture deleted	picture deleted	picture deleted

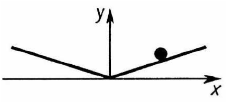
Non-smooth Models: Further Details

Areas: Control, biology, economics, material science, ...

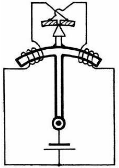
Phenomena: Saturation, "good" numerical behavior, switchings...

Formalisms: differential inclusions, Moreau's sweeping process, ...

Similarities: $m\ddot{x} + h \cdot \text{sign}(x) = 0$



sliding pendulum



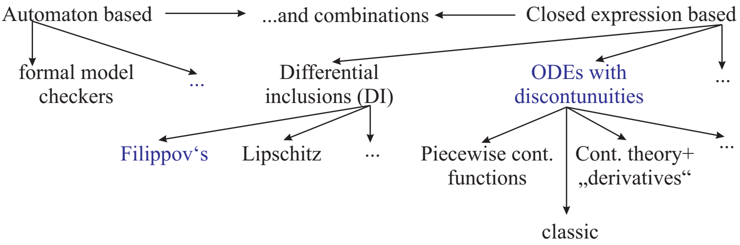
relay

Figures from K. Magnus, 2008: *Schwingungen*, Vieweg+Teubner (in German)

Non-smooth Models: Code Angle

Construct	Example
`IF-THEN-ELSE`	Force: $F \leq 0$
`SWITCH`	Muscle activation function: $0 \leq a(t) = A_1 e^{-c_1(t-t_1)} + A_2 e^{-c_2(t-t_2)} \leq 1$
$\lvert x \rvert$	Hysteresis: $\dot{\omega}(t) = \rho \cdot \Big(v(t) - \sigma \cdot \lvert v(t) \rvert \cdot \lvert \omega(t) \rvert^{\nu-1} \cdot \omega(t) + (\sigma - 1) \cdot v(t) \cdot \lvert \omega(t) \rvert^{\nu} \Big)$
sign$x$	Friction: $F(v) = \text{sign}(v) \cdot F + \mu \cdot v$

Mathematical Formalisms



Necessary: A study of relationships between modeling concepts*

Example: a system with Coloumb and viscous friction

Model: $m\ddot{q}(t) + c\dot{q}(t) + kq(t) \in -\partial(\mu|\dot{q}|)$

Formalisms: DI, variational inequality, interval, ...

* Acary&Brogliato, 2007: *Numerical Methods for Nonsmooth Dynamical Systems*

Possible Interval Reformulations: Example

$\dot{x}(t) = x(t)u(t), x(0) = 0, u(t) \in [-1, 1]$ smooth

Possibility	Formulation	Solution
1. Solve the IVP	see above	$x(t) = 0 \cdot e^{\left(\int_0^t u(s)ds\right)} = 0$
2. Consider a DI	$\dot{x}(t) \in [-x(t), x(t)]$	$x(t) = 0$ and
		$x(t) = \begin{cases} 0, & 0 \le t \le 2 \\ t^2, & t \ge 2 \end{cases}$
3. Use intervals	$\dot{x}(t) = x(t) \cdot [-1, 1]$	$x(t) = 0 \cdot e^{[-1,1]t} = 0$

Systematize formalisms/applications, assign a verified method, introduce a simple way of analysis

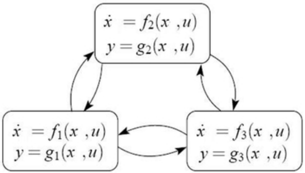
Verified Methods for Non-smooth Systems

Description of a non-smooth IVP

↙ Closed expressions

↘ Automata

$$x' = \begin{cases} f^-(x), & h(x(t),t) < 0 \\ f^+(x), & h(x(t),t) > 0 \end{cases}$$



Rihm (1993), *Mahmoud and Chen (2008)*

Rauh (2006), *Eggers (2008), Ratschan (2012)* Nedialkov and Mohrenschildt (2002)

Verified non-smooth optimization: Slopes, generalized gradients ...

Ratz (1995), Kearfott (2004), Schnurr (2007), ...

Task: Solve the Non-smooth IVP

$\dot{x} = f(x,t),\ x(0) = x_0$, where $f(x,t)$ is non-smooth in x (or in t).

Situation 1: f is discontinuous only in t

$\downarrow$

Lebesque integration, $x(t) = x_0 + \int_0^t f(x(s))ds$

Situation 2: f is discontinuous in x: more difficult

$\downarrow$

- Problem reformulation
- Solution definition (allowed to be discont.?)
- Existence (uniqueness) of the solution
- Application areas

Rihm's method

Reformulation $\dot{x}(t) = f(x(t), t) = \begin{cases} f_1(t, x(t)), g(t, x(t)) < 0 \\ f_2(t, x(t)), g(t, x(t)) > 0 \end{cases}$

Application systems with friction, switchings

Solution
(a) a continuous function for which IVP holds except on isolated switching points
(b) Filippov's convex definition for DIs if switching points are not isolated

Existence
(a) unique if transversality conditions hold
(b) exists if the function $f_0 = \alpha \cdot f_1 + (1 - \alpha) f_2$ is cont. in x; unique if f_0 cont. differentiable

Rihm's method (Cont.)

Transversality

$$\dot{g}_1(t,x) := \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f_1(t,x) > 0 (< 0)$$

$$\dot{g}_2(t,x) := \frac{\partial g}{\partial t} + \frac{\partial g}{\partial x} f_2(t,x) > 0 (< 0)$$

Sliding $\dot{g}_1(t,x) < 0$ and $\dot{g}_2(t,x) > 0$

Endpoints $\dot{g}_1(t,x) > 0$ and $\dot{g}_2(t,x) < 0$

Let t^* be the switching point, $x^* := x(t^*)$, $f^- := f_1(t^-, x(t^-))$, $f^+ := f_2(t^*, x^*)$, $h^- := t^* - t^-$, $h^+ := t^+ - t^*$, $s = t^+ - t^-$, $z^- \in \mathbf{z}^-$ and $z^+ \in \mathbf{z}^+$ local errors, then

$$\begin{aligned} x(t^+) &= x(t^-) + h^+(f^+ - f^-) + s f^- + z^- + z^+ \\ &\in \mathbf{x}^- + s\mathbf{f}^- + [0,s](\mathbf{f}^+ - \mathbf{f}^-) + \mathbf{z}^- + \mathbf{z}^+ =: \mathbf{x}^+ \end{aligned}$$

An Automaton Based Method (Rauh et al.)

Problem: Smooth models $\{S_i\}_{i=1}^{l}$: $\dot{x}(t) = f_{S_i}(x(t), p, u(t), t)$
Transition $S_i \rightarrow S_j$ if the transition $T_i^j(x, u)$ holds true

Stage 1: Calculate a bounding box

$$\mathbf{b}_k^a = \bigcup_{i \in \mathcal{I}_a} (\mathbf{x}_0 + [0, h] \cdot f_{S_i}(\mathbf{x}_k, \mathbf{p}, \mathbf{u}(t_k), t_k))$$

Stage 2: Activate additional transitions $T_i^j(\mathbf{b}_k^a, \mathbf{u}[t_k, t_{k+1}])$

$$\tilde{\mathbf{b}}_k^a := \mathbf{b}_k^a \bigcup_{i \in \tilde{\mathcal{I}}_a \setminus \mathcal{I}_a} (\mathbf{x}_0 + [0, h] \cdot f_{S_i}(\mathbf{b}_k^a, \mathbf{p}, \mathbf{u}([t_k, t_{k+1}], [t_k, t_{k+1}])))$$

Stage 3: Calculate $\mathbf{x}_{k+1}$ at t_{k+1} (Taylor/Euler wrt smoothness)
Stage 4: Deactivate transition conditions depending on $\mathbf{x}_{k+1}$

A "Continuous" Method: Problem Definition

Interval IVP: $\begin{cases} x' = f(x), \\ x(0) \in [x_0] \end{cases}$

where $f : \mathcal{D} \subset \mathbb{R}^n \to \mathbb{R}^n$ or $\mathcal{D} \subset \mathbb{IR}^n \to \mathbb{IR}^n$ and is given in algorithmic representation:

$$\begin{cases} \tau_i(x) = g_i(x) = x_i,\ i = 1 \dots n \\ \tau_i(x) = g_i(\tau_1(x), \dots, \tau_{i-1}(x)),\ i = n+1 \dots l, \\ g_i \in S_{EO} \cup S_{PW} \end{cases} .$$

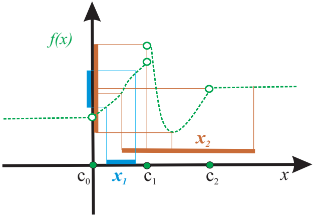
$S_{EO} = \{+, -, *, /, \sin, \cos, \dots\}$ and S_{PW} are piecewise cont.

Goal: Find a derivative generalization to use with the usual theory

Definition of Piecewise Functions $\phi(y)$ in S_{PW}

$y = \tau_\nu(x)$, $\phi_j(y)$, $j = 0, \dots, L$ smooth

$$\phi(y) = \begin{cases} \phi_0(y) & \text{for } c_{-1} = -\infty < y < c_0, \\ \phi_1(y) & \text{for } c_0 < y < c_1, \\ \dots & \dots \\ \phi_{L-1}(y) & \text{for } c_{L-2} < y < c_{L-1}, \\ \phi_L(y) & \text{for } c_{L-1} < y < c_L = +\infty \;. \end{cases}$$



An interval extension of ϕ over $\mathbf{x}$ ($\phi(\mathbf{x})$):

$$\begin{cases} \phi_i(\mathbf{x}), & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_i), \\ \underline{\bigcup}_{k=i+1}^{j-1} \phi_k([c_{k-1}, c_k]) \underline{\cup} \phi_i([\underline{x}, c_i]) \underline{\cup} \phi_j([c_{j-1}, \overline{x}]), & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_j) \end{cases}$$

Definition of the Derivative

An interval extension of ϕ' over $\mathbf{x}$ ($\phi'(\mathbf{x})$)

$$\begin{cases} \phi_i'(\mathbf{x}), & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_i), \\ \underline{\bigcup}_{k=i+1}^{j-1} \phi_k'([c_{k-1}, c_k]) \underline{\cup} \phi_i'([\underline{x}, c_i]) \underline{\cup} \phi_j'([c_{j-1}, \overline{x}]) & \\ \underline{\cup}\ \mathrm{REST}, & \text{if } \mathbf{x} \subseteq (c_{i-1}, c_j), \end{cases}$$

where REST depends on:

- how many switching points $\mathbf{x}$ contains,
- whether ϕ is continuous,

if we want the mean value theorem to hold.

Derivative for IF-THEN-ELSE (One Switching Point)

$$\phi(x) = \begin{cases} \phi_0(x), & x < c_0, \\ \phi_1(x), & x > c_0. \end{cases}$$

then REST is

- $(\phi_0'([\underline{x}, c_0]) + \phi_1'([c_0, \overline{x}])) \cdot [0, 1]$ if ϕ is continuous,
- $$\left(\frac{\phi_1(c_0) - \phi_0(c_0)}{[c_0, \overline{x}] - x_0} + (\phi_0'([\underline{x}, c_0]) + \phi_1'([c_0, \overline{x}])) \cdot [0, 1] \right)$$
  $$\underline{\cup} \left(\frac{\phi_0(c_0) - \phi_1(c_0)}{[\underline{x}, c_0] - x_0} + (\phi_0'([\underline{x}, c_0]) + \phi_1'([c_0, \overline{x}])) \cdot [0, 1] \right)$$
  if ϕ is discontinuous.

x_0 needed to avoid overconservative enclosures ($[-\infty, +\infty]$)

Generalization

This definition can be generalized for the arbitrary number of $c_i \in \mathbf{x}$, but:
$\mathbf{x}$ containing many c_i might be simply too wide.

If $\phi(\cdot)$ contains several switching points c_i, but $\mathbf{x}$ contains only one:

$$\phi'(\mathbf{x}) = \begin{cases} \phi_i'(\mathbf{x}) & \text{for } \mathbf{x} \subset (c_{i-1}, c_i), \\ \phi_{\text{cont}}'(\mathbf{x}) & \text{for } \mathbf{x} \subset (c_{i-1}, c_{i+1}), \text{ if } \phi \text{ is cont. in } c_i, \\ \phi_{\text{dis}}'(\mathbf{x}) & \text{for } \mathbf{x} \subset (c_{i-1}, c_{i+1}), \text{ if } \phi \text{ is discont. in } c_i \end{cases}.$$

Features

+	Right sides with several variables represented	$f(x_1, x_2) = \lvert x_1 \rvert + x_1 \cdot \text{sign}\,(x_2)$, $\lvert\text{sign}\,(x_1)\rvert$
−	No PW operations like	$f(x_1, x_2) = \begin{cases} x_1, & x_2 < 0 \\ x_1 \cdot x_2, & x_2 > 0 \end{cases}.$
	Covered by Rihm	$g(x_1, x_2) = x_2 - 0$, $f_1 = x_1 \;\; f_2 = x_1 \cdot x_2$
+	Better coverage for	$f(x) = \begin{cases} -h, & x < -x_+ \\ 0, & -x_+ < x < x_+ \\ h, & x_+ < x \end{cases}.$

Usage

Approach Plug the definition into VALENCIA-IVP

VALENCIA

- an a posteriori method
- for smooth problems
- $x(t) \in \underbrace{[x(t)]}_{\text{verified enclosure}} := \underbrace{x_{app}(t)}_{\text{approximation}} + \underbrace{[R(t)]}_{\text{error bounds}}$
- uses MVT, a fixed point theorem
- Jacobians only → easy to adapt

\+ for Lipschitz cont. right sides or isolated switching points

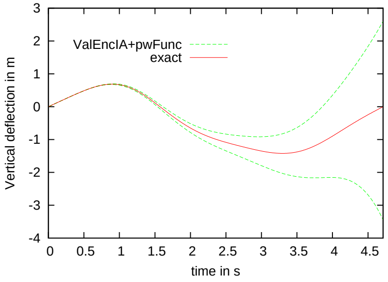
– overestimation for sliding solutions

Known Exact Solution: Tacoma Narrows Suspension Bridge

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0$$

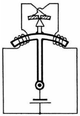
$$\dot{x}_2 = \frac{1}{m}\left(\sin(4t) - q(x_1)\right) \qquad x_2(0) = 1$$

$$q(x_1) = \begin{cases} x_1, & x_1 < 0 \\ 4x_1, & x_1 > 0 \end{cases}$$

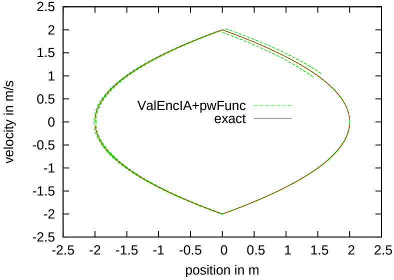


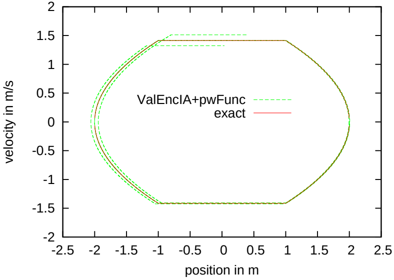
Known Exact Solution: Oscillator

$$m(\ddot{x}) = -f_i(x),\ h = m = x_t = 1,\ x(0) = 2,\ v(0) = 0$$



$$f_1(x_1) = \begin{cases} -h, & x < 0 \\ +h, & x > 0 \end{cases} \qquad f_2(x_1) = \begin{cases} -h, & x < -x_t \\ 0, & -x_t < x < x_t \\ +h, & x > x_t \end{cases}$$



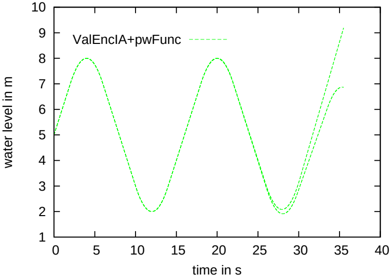


Comparisons with Other Methods: Water Level

$$\dot{x}_1 = x_2 \qquad x_1(0) = 5$$
$$\dot{x}_2 = 0.5u(x_1) \qquad x_2(0) = 1$$

$$u(x_1) = \begin{cases} 1, & x_1 < 3 \\ -1, & x_1 > 7 \\ 0, & otherwise \end{cases}$$

Width at $t = 35$: wid$(\mathbf{x}_1) = 0.28$ as opposed to Nedilkov/von Mohrenschildt wid$(\mathbf{x}_1) = 10^{-7}$



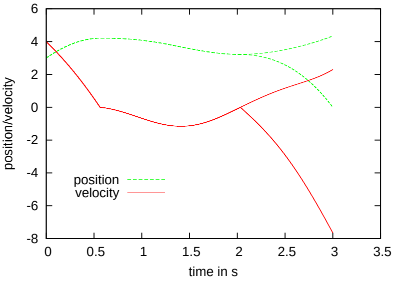
Example without a Classical Solution

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -0.2x_2 - x_1 + 2\cos(\pi t) - u(x_2)$$

$$u(x_1) = \begin{cases} -4, & x_1 < 0 \\ +4, & x_1 > 0 \end{cases}$$

$$x_1(0) = 3, x_2(0) = 4$$

The first switching point at $t \approx 0.5$, the second at $t \approx 2.03$, the solution leaves the switching surface $v = 0$ after $t \approx 2.6$



Applications: System with Friction and Hysteresis (1)

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \frac{1}{m}\left(F_a(t) - F_f(x_2)\right) \end{bmatrix} \quad x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

Friction force:

$$F_f(x_2) = \begin{cases} -[F_s] + [\mu] \cdot x_2 & \text{for} \quad S_1 = \texttt{true} \quad \text{or} \quad S_2 = \texttt{true} \\ +[F_s] + [\mu] \cdot x_2 & \text{for} \quad S_4 = \texttt{true} \quad \text{or} \quad S_5 = \texttt{true} \end{cases}$$

with the static friction

$$F_f(x_2) \in [F_s^{max}] := \left[-\overline{F}_s \,;\, \overline{F}_s\right] \quad \text{for} \quad S_3 = \texttt{true}$$

$$S_1 = \{x < 0, \omega \geq 0\},\ S_2 = \{x < 0, \omega < 0\},\ S_3 = \{x = 0\},$$
$$S_4 = \{x > 0, \omega \geq 0\},\ S_5 = \{x < 0, \omega > 0\}$$

Applications: System with Friction and Hysteresis (2)

Accelerating force:

$$F_a\,(t) := u\,(t) - \phi\,(x_1\,(t)\,, \omega\,(t))$$

- control variable $u\,(t)$ provided by an actuator
- restoring spring force $\phi\,(x_1\,(t)\,, \omega\,(t)) = \kappa_x x_1 + \kappa_\omega \omega$

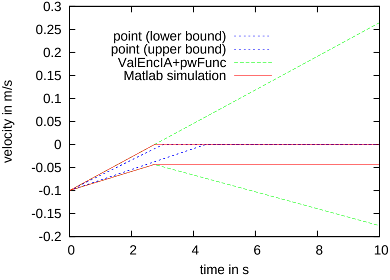
Restoring spring force with hysteresis (the Bouc-Wen model):

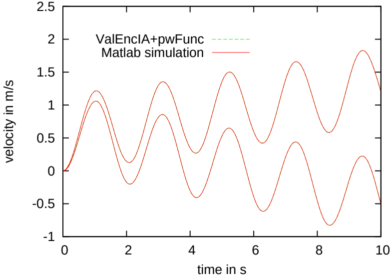
$$\begin{aligned}\dot{\omega}\,(t) =& \rho \cdot \Big(x_2\,(t) - \sigma \cdot |x_2\,(t)| \cdot |\omega\,(t)|^{\nu-1} \cdot \omega\,(t) \\ &+ (\sigma - 1) \cdot x_2\,(t) \cdot |\omega\,(t)|^{\nu}\Big)\end{aligned}$$

Applications: System with Friction and Hysteresis (3)

$$\kappa_x = 0$$
$$x_1(0) = 0$$
$$x_2(0) = -0.1$$
$$\omega(0) = -0.001$$
$$u(t) = 0.01$$

$$\kappa_x = 0.001$$
$$x_1(0) = 0$$
$$x_2(0) = 0$$
$$\omega(0) = 0.001$$
$$u(t) = 2\sin(3t)$$





Applications: Sliding Mode Control for SOFC (1)

Goal:

- Avoid too high temperatures
- Keep ϑ_{FC} at $\vartheta_{FC,d}$ for all $p \in [p]$, $d \in [d]$

Equations:

$$\dot{\vartheta}_{FC} = a(\vartheta_{FC}(t), p, d) + b(\vartheta_{FC}(t), p, d) \cdot u(t)$$

Stabilization:

$$V(s(t)) = \frac{1}{2}s(t)^2 > 0,$$

$$\dot{V}(s(t)) = s(t) \cdot \dot{s}(t) \leq -\eta \cdot |\vartheta_{FC}(t) - \vartheta_{FC,d}|$$

Control:

$$[u] := \frac{-a(\vartheta_{FC}(t), [p], [d]) - \eta \cdot \mathrm{sign}(\vartheta_{FC}(t) - \vartheta_{FC,d})}{b(\vartheta_{FC}(t), [p], [d])}$$

Applications: Sliding Mode Control for SOFC (2)

Used non-smooth functions:

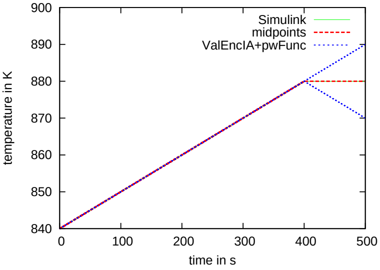
- sign(ϑ)
- different control laws for $\vartheta \lessgtr 800$

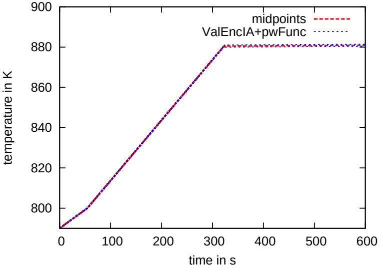
Simplification:

constant parameters

Conditions:

1. $\vartheta_{FC}(0) = 840$K, no disturbance
2. $\vartheta_{FC}(0) = 790$K, $d(t) = 0.1\text{sign}(\sin(0.001t))$





Conclusions

Results:

- Implementation of a simple extension for non-smooth functions in ValEncIA
- Analysis of the method
- Application for systems with friction, sliding mode control in SOFC

Future work:

- Application to stance stabilization
- Reduction of overestimation (slopes?)
- Detection and special handling for sliding
- Relations of general formalisms and interval formulations