Rogue waves statistics in the framework of one-dimensional Generalized Nonlinear Schrodinger Equation.

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1. Introduction.

The main purpose of this work is the investigation of the frequency of occurrence of high waves (rogue waves) in the framework of Generalized one-dimensional Nonlinear Schrodinger Equation.

There are different definitions of rogue waves. For example, for oceanic waves studies the following criteria is frequently used:

$$H / H_s > 2,$$

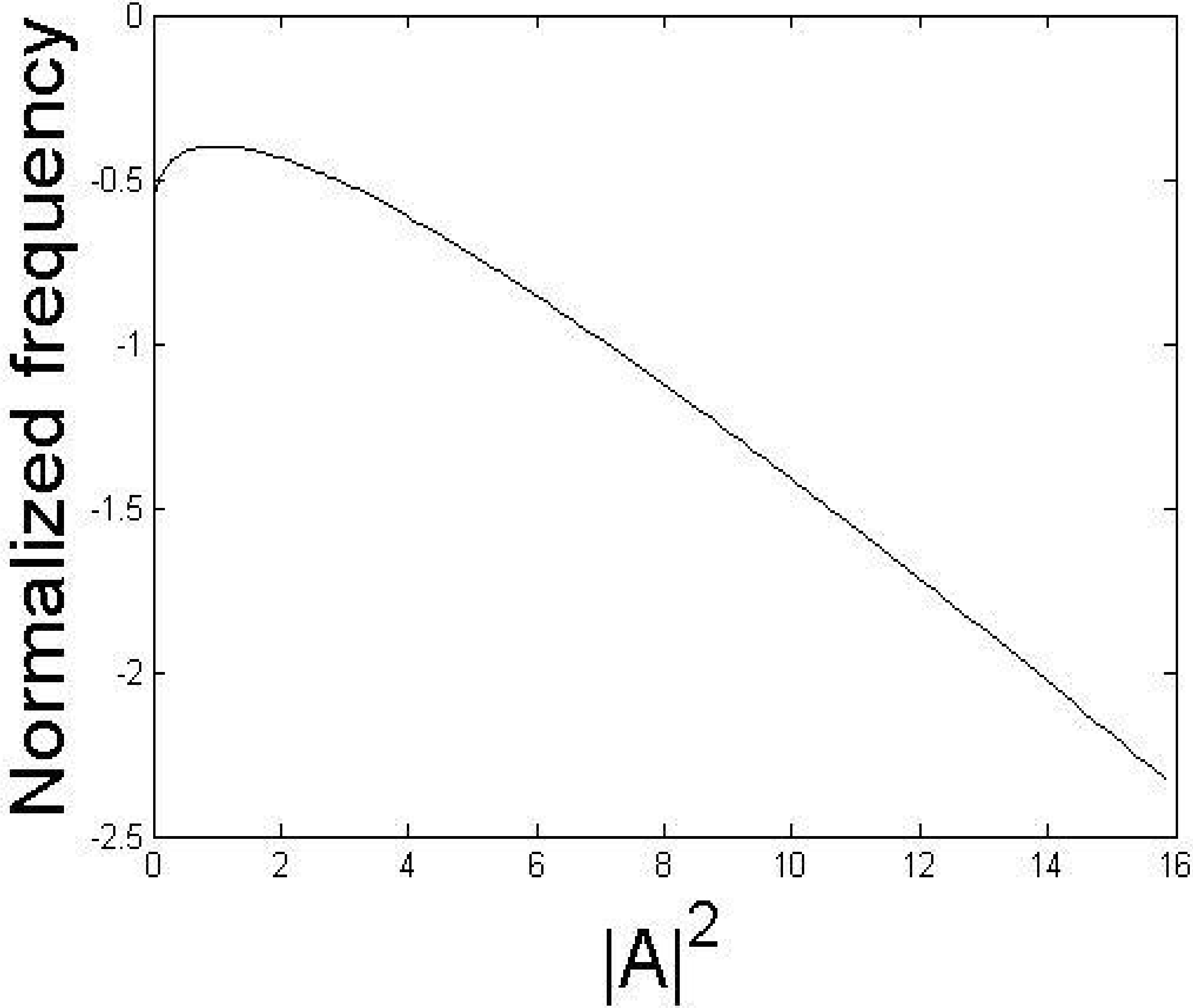
where **H** is wave height (distance from trough to crest), and $\mathbf{H_s}$ is the significant wave height defined as four times the standard deviation of the surface elevation (see for example K.Dysthe, H.E.Krogstad, P.Muller, Oceanic Rogue Waves, Annu. Rev. Fluid Mech. 2008. 40:287-310).

Let us suppose that the current state of a system consists of multitude of uncorrelated linear waves:

$$\psi = \sum_k a_k \, exp(kx - \omega_k t + \phi_k).$$

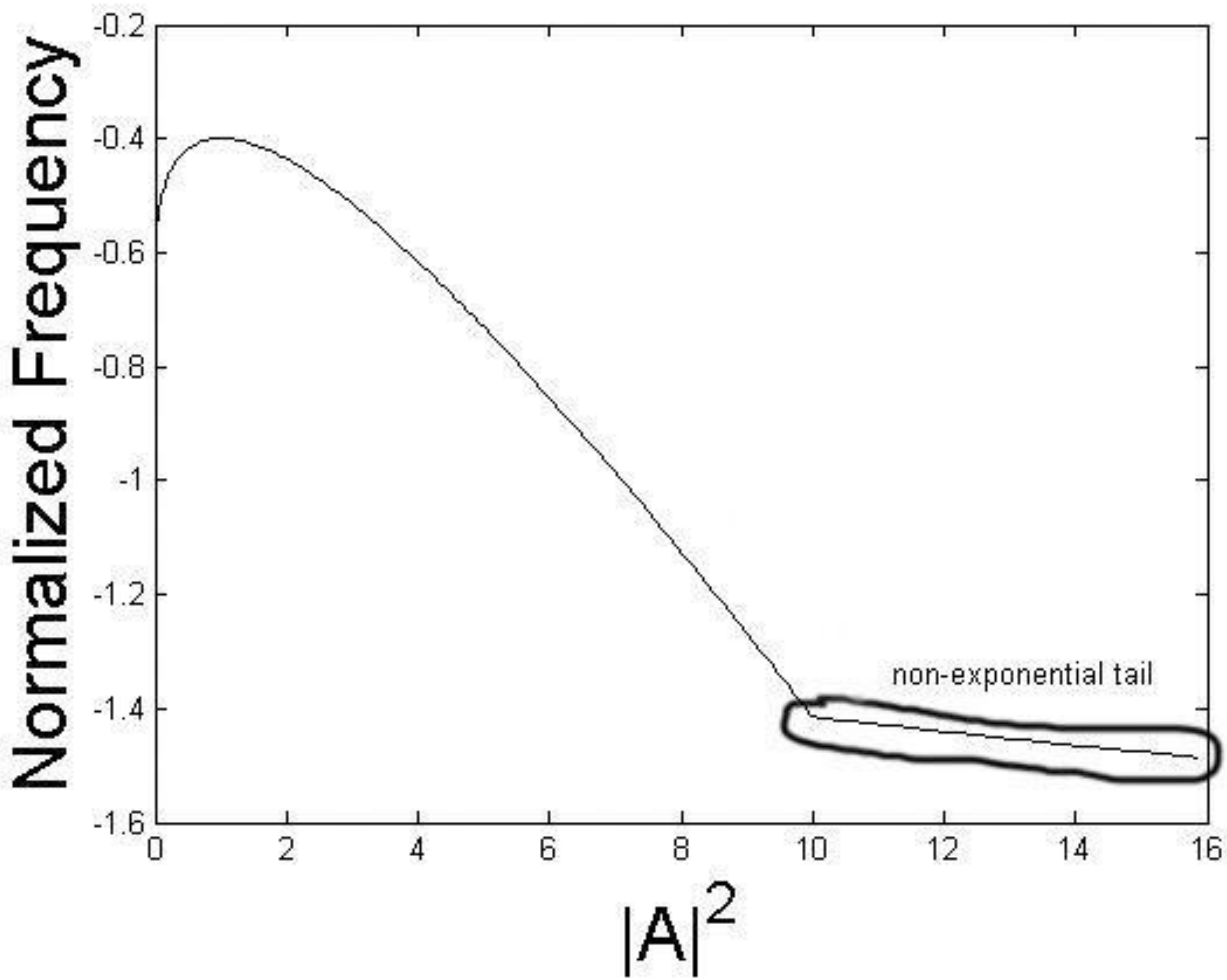
When $\mathbf{a_k}$ and $\boldsymbol{\varphi_k}$ are random uncorrelated values and number of linear waves (**k**) is large enough, $\boldsymbol{\psi}$ is distributed according to Gaussian distribution:

$$P(\psi) \propto exp(-|\psi|^2 / 2\sigma^2).$$



0
-0.5
-1
-1.5
-2
-2.5
0
2
4
6
8
10
12
14
16
Normalized frequency
|A|2

In this work we search for deviations of probability density functions (PDFs) for waves amplitudes **ψ** from Gaussian distribution when large waves appearance frequency turns out to be much higher than predicted by the linear theory:



The sequence of our research can be described as follows:

1) We take a nonlinear system - classical integrable NLS equation of focusing type or generalized NLS equation with dynamics close to that of the classical NLS equation of focusing type.

2) We perform numerical simulation of this system for initial distribution **ψ = 1 + noise** with fixed statistical properties of noise up to some pre-defined time-shift **ΔT**.

3) We perform 10 000 of such simulations, that differ from each other only by noise realizations.

4) Modulation instability generates peaks which for large waves are close to solitons of the classical NLS equation. Therefore, for large waves we have in fact solitonic turbulence.

5) We measure probability density function (PDF) of waves appearance on the basis of our ensemble of 10 000 simulations.

6) We examine how PDFs depend on noise properties, time-shift **ΔT** and coefficients before the additional terms beyond the classical NLS equation.

2. Nonlinear systems.

We examine the following nonlinear systems:

(1) (Conservative) classical NLS equation of the focusing type:

$$i\psi_t + \psi_{xx} + |\psi|^2 \psi = 0;$$

(2) (Conservative) generalized NLS equation accounting for 6- and 8-wave interactions:

$$i\psi_t + \psi_{xx} + |\psi|^2 \psi + \alpha |\psi|^4 \psi - \beta |\psi|^6 \psi = 0, \quad \alpha << 1, \quad \beta << \alpha;$$

(3) (Non-conservative) generalized NLS equation accounting for 6-wave interactions, linear and nonlinear dumping terms as well as a pumping term:

$$i\psi_t + (1 - ia)\psi_{xx} + |\psi|^2 \psi + (\alpha + ib)|\psi|^4 \psi = ic\psi, \quad \alpha, a, b, c << 1, \quad b << \alpha;$$

The difference between the systems (2) and (3) is in specific way of regularization of the collapsing term $|\boldsymbol{\psi}|^4 \boldsymbol{\psi}$. Coefficients before the additional terms to NLS equation are small, so that the dynamics of the systems (2) and (3) is close to that of the classical NLS equation (1).

For numerical simulations we used Split-Step method of the second order in time with adaptive change of spatial grid size Δx. The computational domain was $[-16\pi, 16\pi)$ (modulation instability generates 16 peaks) with periodic boundary conditions. Time step Δt and grid size Δx were linked with the relation $\Delta t = q_0 \Delta x^2$, where $q_0 < 0.1$ (see for instance T.I. Lakoba, Stability analysis of the split-step Fourier method on the background of a soliton of the nonlinear Schrodinger equation, arXiv: 1008.4974v1).

We checked our results using the same Split-Step 2nd order method with $q = 0.25q_0$, and also with the help of Split-Step 4th order, Runge-Kutta 4th and 5th order methods.

It turns out that for the considered systems results for a *single simulation* coincide ($D(\psi_1, \psi_2) = ||\psi_1 - \psi_2|| / ||\psi_2|| << 1$) up to time-shifts $\Delta T \sim 100$. Beyond $\Delta T \sim 100$ difference between the results becomes comparable with unity.

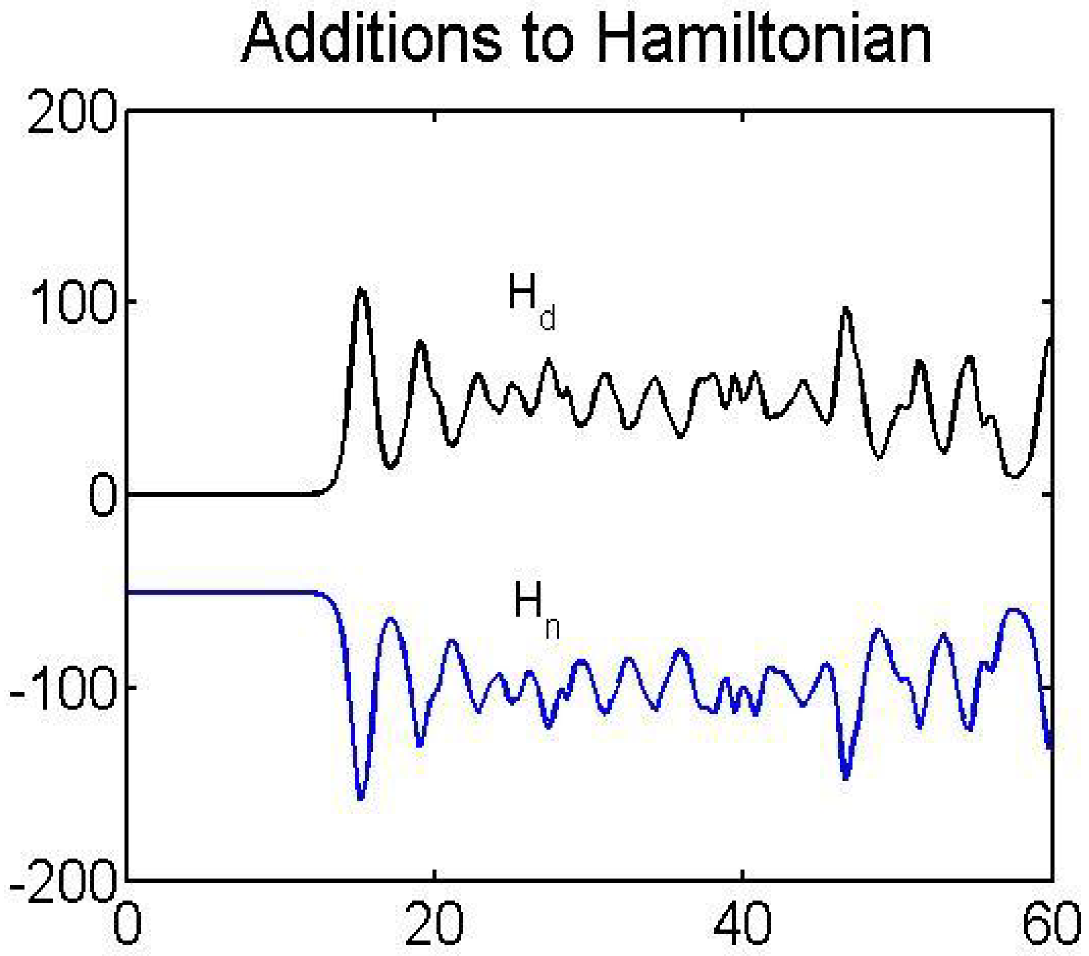
For instance, all the numerical methods we used give the same results for the classical NLS equation up to $\Delta T_0 = 60$, and all the methods give different results beyond this time-shift due to quasi-periodicity of NLS equation dynamics (Fermi-Pasta-Ulam phenomenon): near ΔT_0 distribution ψ is close again to it's initial stage **1 + noise**, but this time noise contains different errors for different numerical methods.

Nevertheless, we found that the *statistical results*, in particular PDFs, do not depend on specific numerical method we used, at least up to our final time-shifts.

2.1. Classical NLS equation.

NLS equation is the Hamiltonian one, where the Hamiltonian can be represented as sum of kinetic (dispersion) and potential (nonlinear) energy:

$$H = H_d + H_n, \quad H_d = \int |\psi_x|^2 \, dx, \quad H_n = -(1/2)\int |\psi|^4 \, dx.$$



NLSE also conserves wave action **N** and momentum **P**:

$$N = \int |\psi|^2 \, dx,$$

$$P = (i/2)\int \left(\psi_x^* \psi - \psi_x \psi^*\right) dx.$$

Evolution of $\mathbf{H_n}$ (blue line) and $\mathbf{H_d}$ (black line) for classical NLS equation.

2.2. Generalized NLS equation accounting for 6- and 8-wave interactions.

$$i\psi_t + \psi_{xx} + |\psi|^2 \psi + \alpha |\psi|^4 \psi - \beta |\psi|^6 \psi = 0, \quad \alpha << 1, \quad \beta << \alpha;$$

We performed full-scale numerical simulations for the following set of coefficients:

(a) α=0.040, β=0.001. Cubic nonlinearity dominates at **|ψ|<5**, 5th-order nonlinearity - at **5<|ψ|<6.3**, 7th-order nonlinearity - at **|ψ|>6.3**.

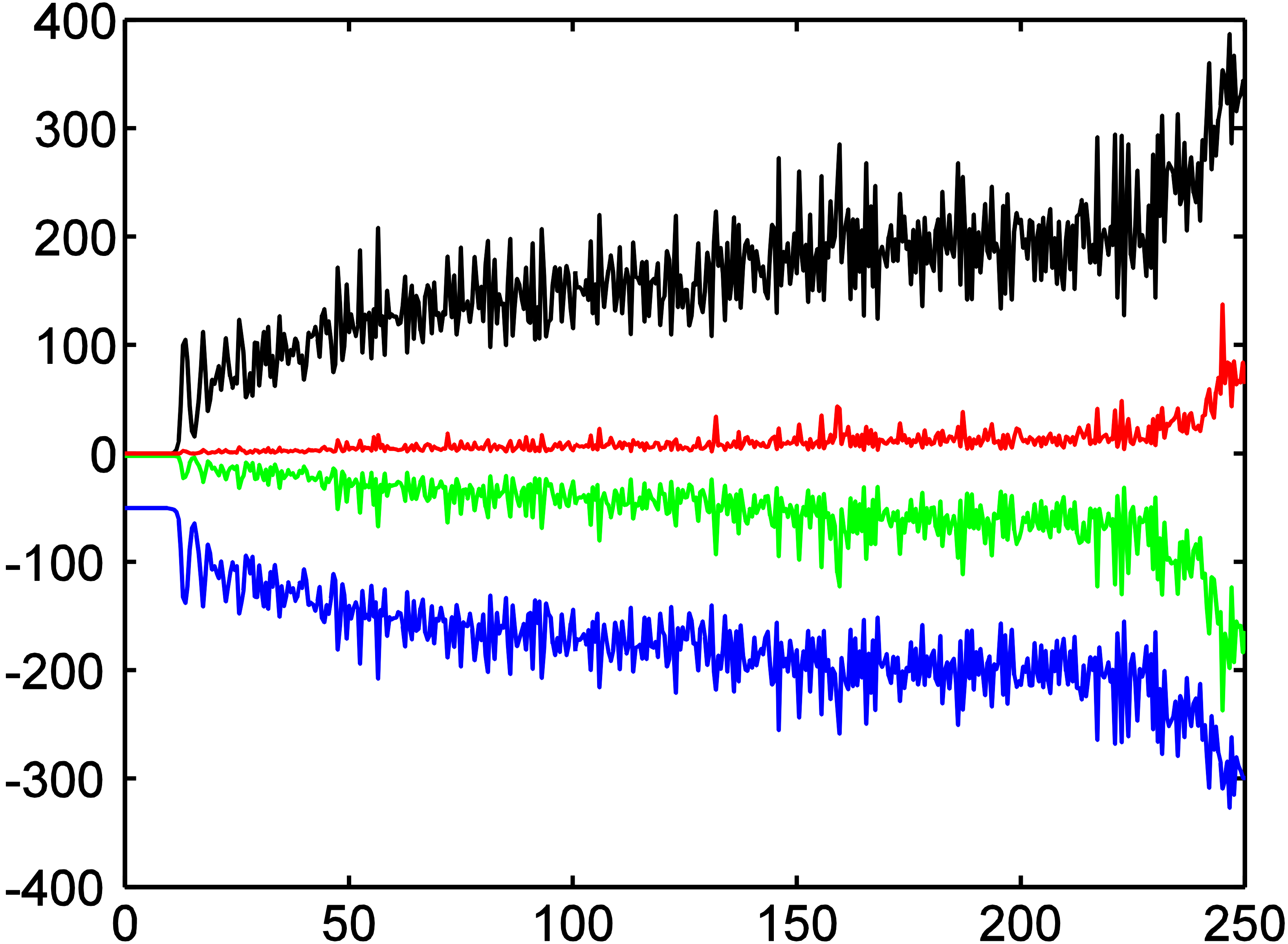
(б) α=0.064, β=0.002. Cubic nonlinearity dominates at **|ψ|<4**, 5th-order nonlinearity - at **4<|ψ|<5.6**, 7th-order nonlinearity - at **|ψ|>5.6**.

Hamiltonian of the system can be represented as follows:

$$H = H_d + H_{n1} + H_{n2} + H_{n3}, \quad H_d = \int |\psi_x|^2 dx, \quad H_{n1} = -(1/2)\int |\psi|^4 dx,$$

$$H_{n2} = -(\alpha/3)\int |\psi|^4 dx, \quad H_{n3} = -(\beta/4)\int |\psi|^6 dx.$$

Additions to Hamiltonian

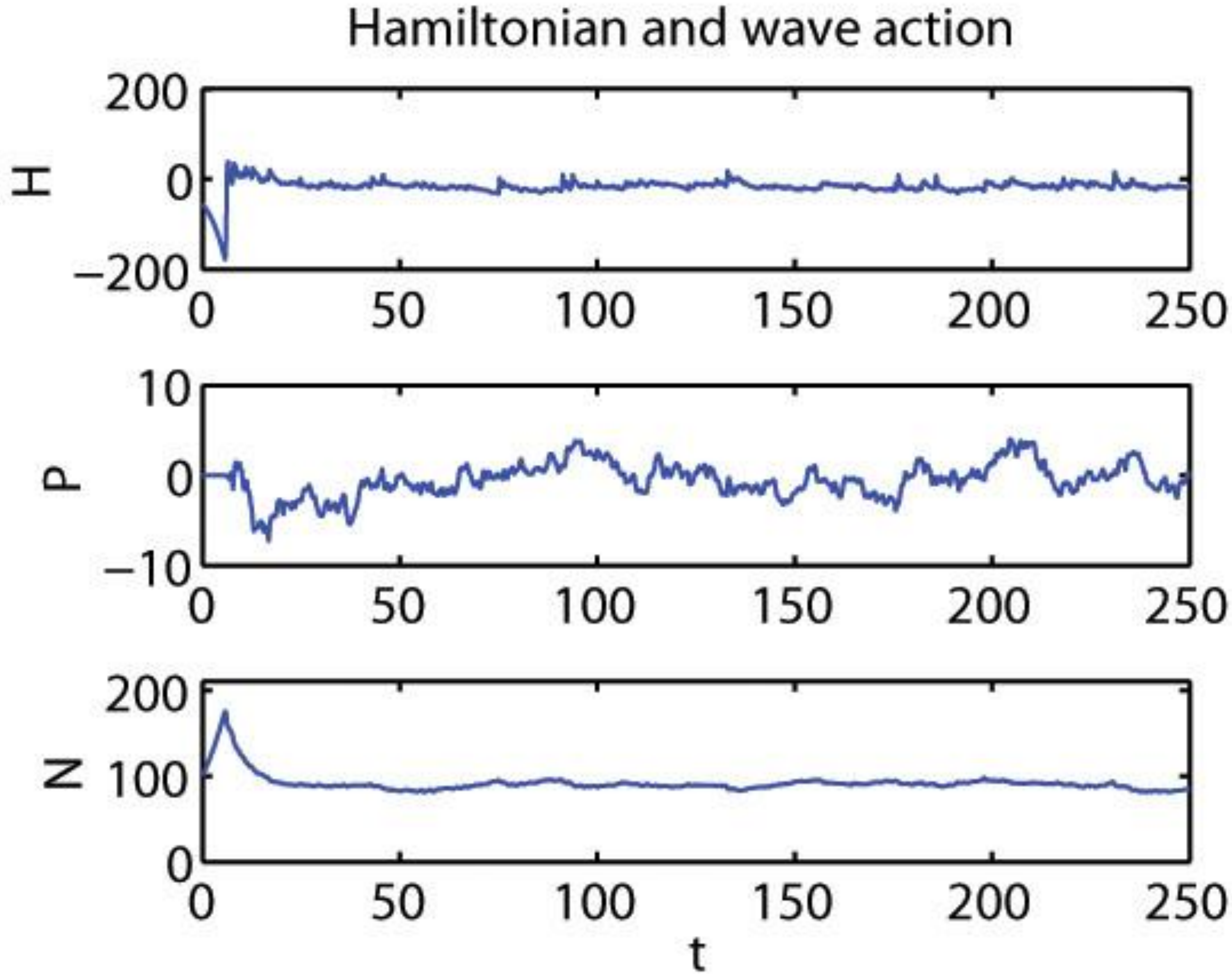


Evolution of H_d (black), H_{n1} (blue), H_{n2} (green), H_{n3} (red).

2.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms.

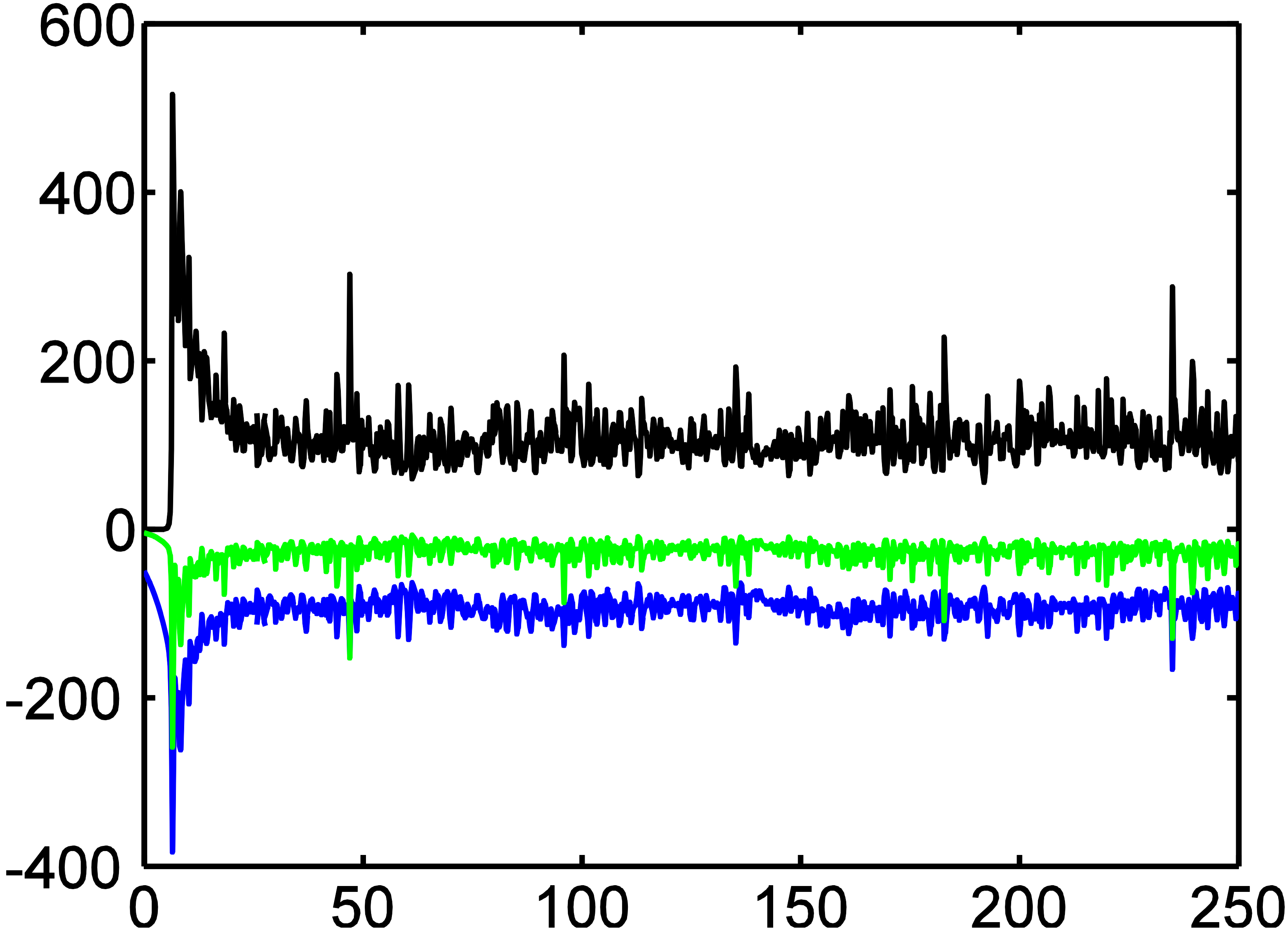
$$i\psi_t + (1 - ia)\psi_{xx} + |\psi|^2 \psi + (\alpha + ib)|\psi|^4 \psi = ic\psi, \quad \alpha, a, b, c << 1, \quad b << \alpha;$$

We performed numerical simulations for **α = 0.032, 0.064, 0.128, 0.256** for different coefficients before dumping (**a, b**) and pumping terms (**c**). For full-scale simulations the following dumping and pumping parameters were used: **a=0.04, b=0.004, c=0.05**.



Evolution of Hamiltonian, momentum, and wave action.

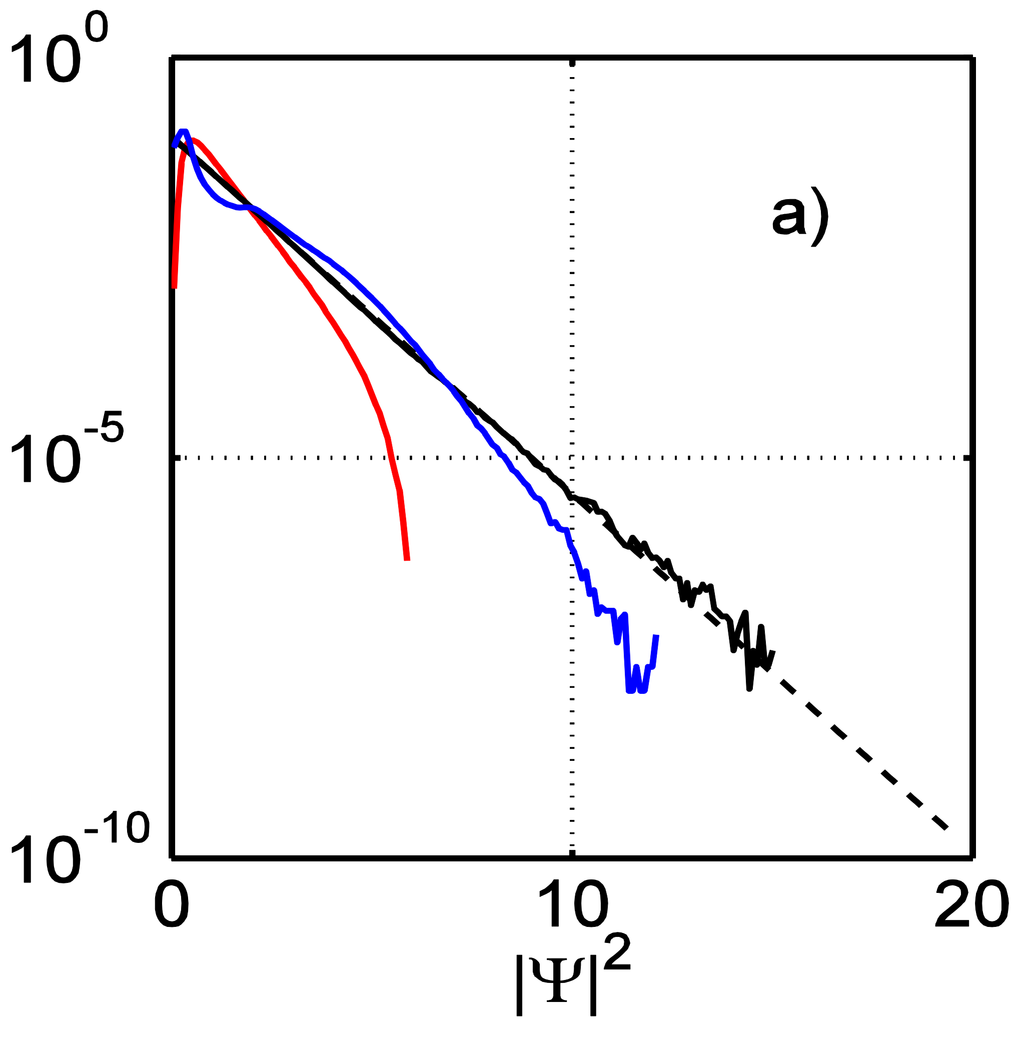
Additions to Hamiltonian



Evolution of H_d (black), H_{n1} (blue), H_{n2} (green).

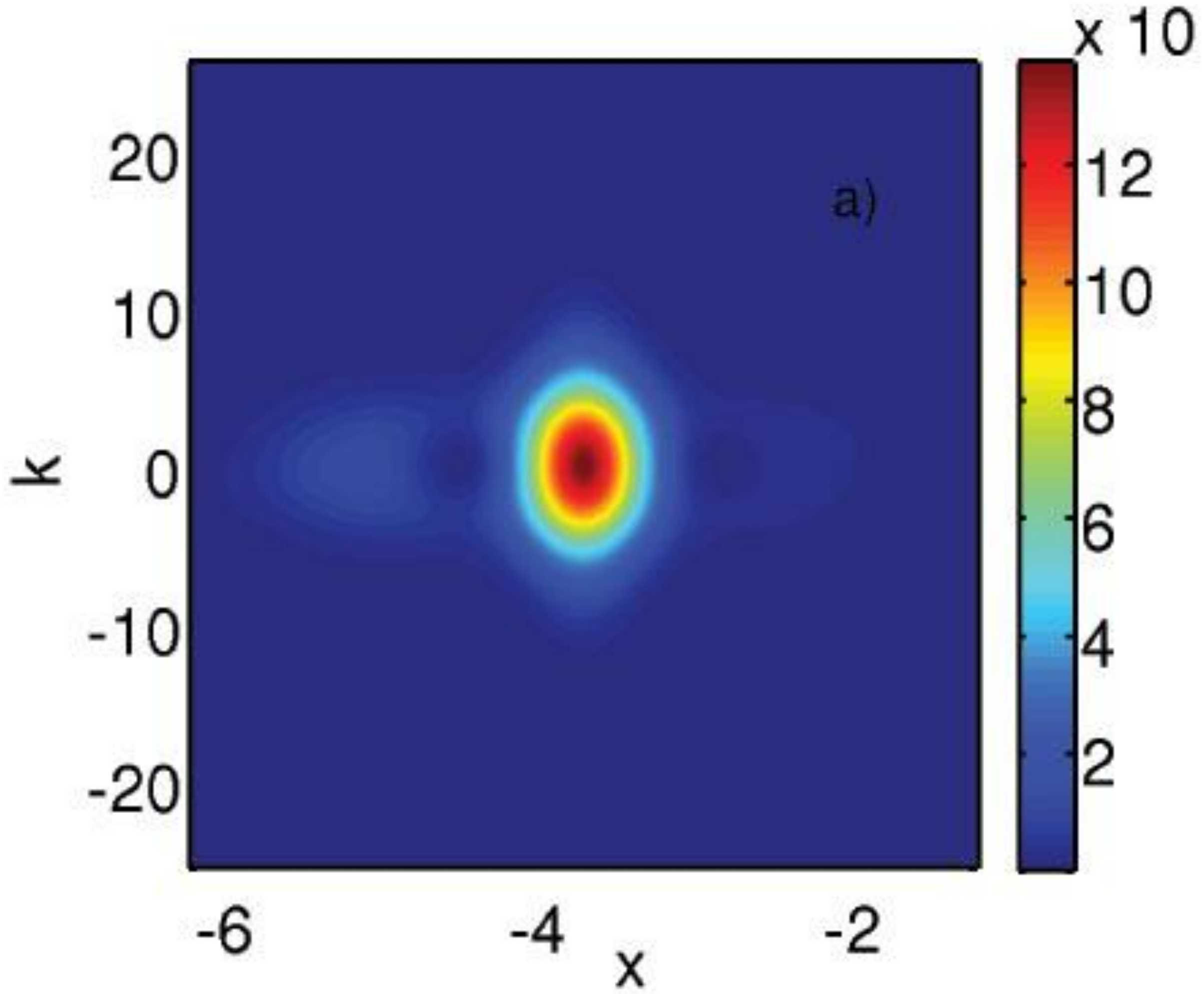
3. Rogue waves statistics.

3.1. Classical NLS equation: Gaussian statistics with the exception of time-shits close to quasi-periodicity points.

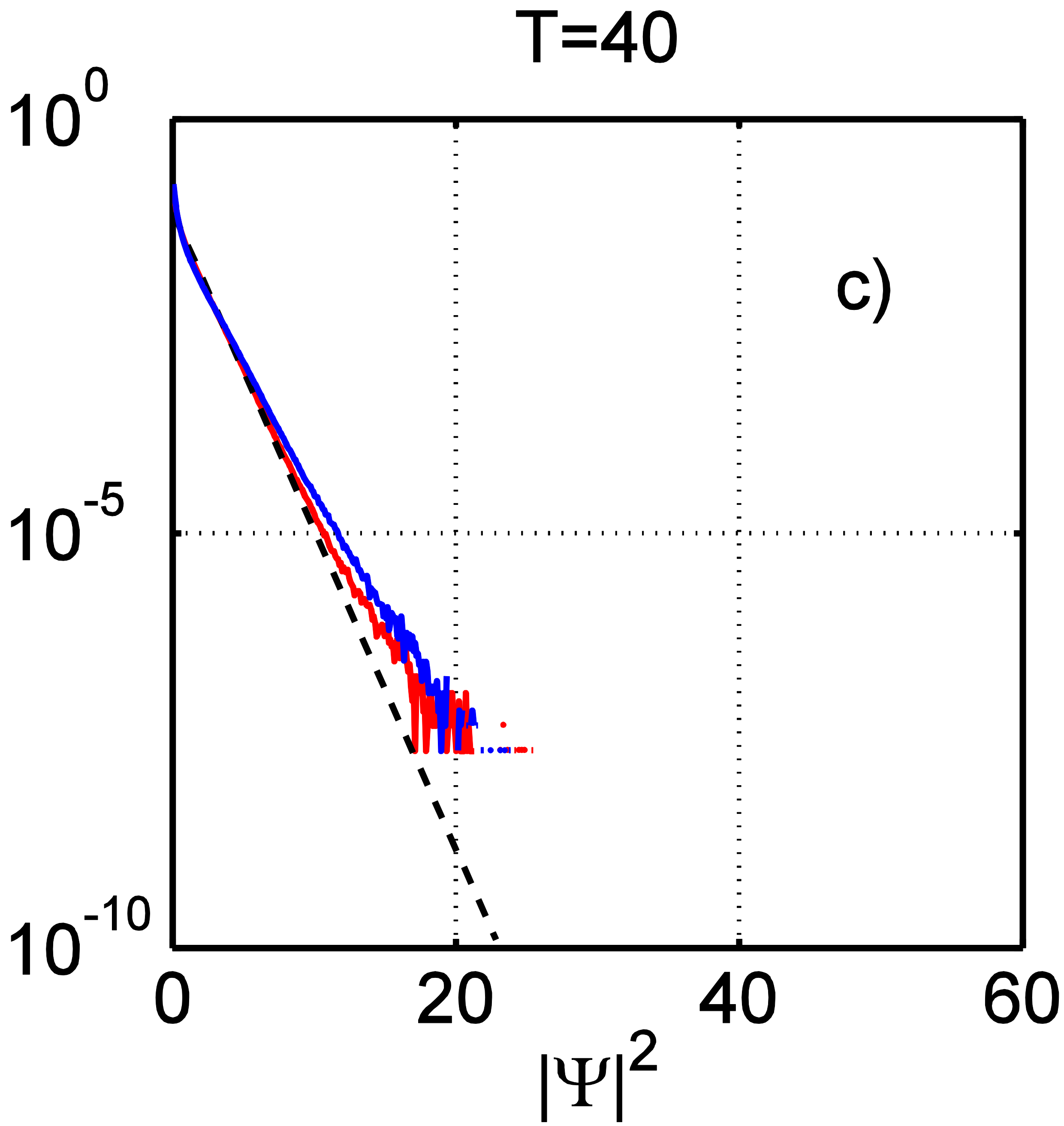


Statistics at time-shifts t=10 (red), t=12 (blue), t=30 (black).

3.1. Classical NLS equation: High waves are collisions of solitons.



3.2. Generalized NLS equation accounting for 6- and 8-wave interactions: initially statistics is close to Gaussian, when non-exponential tails rise.



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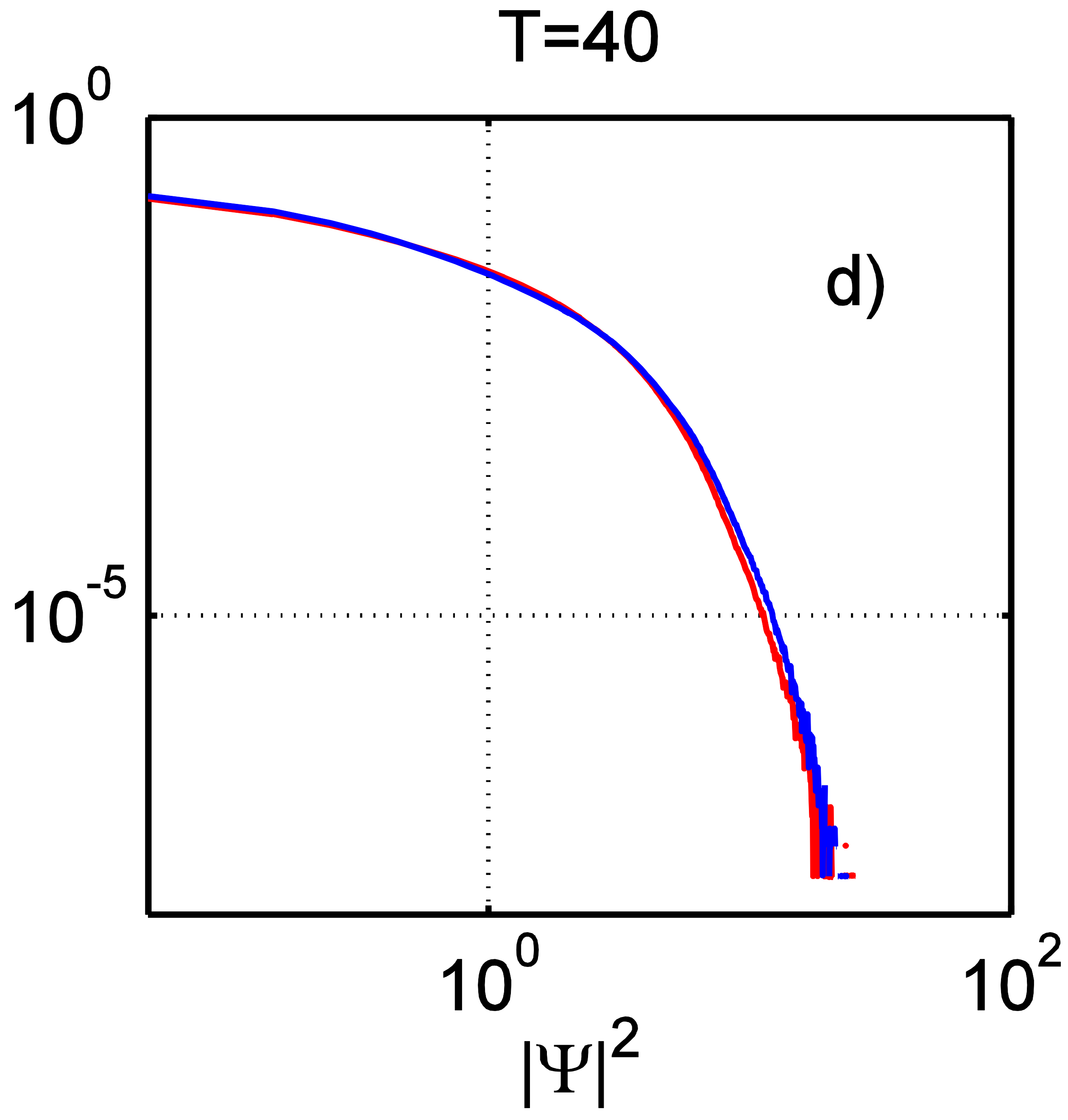

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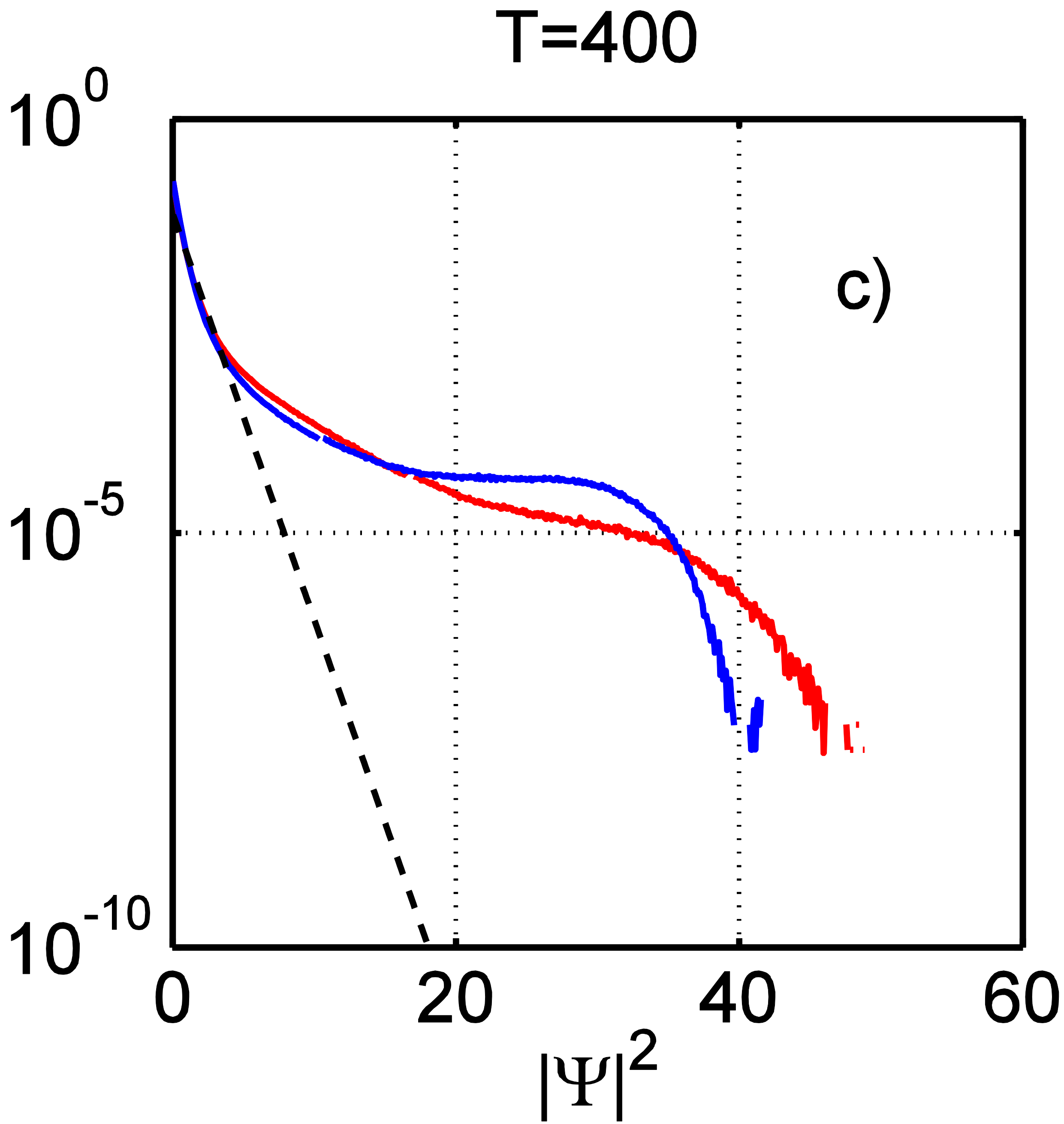

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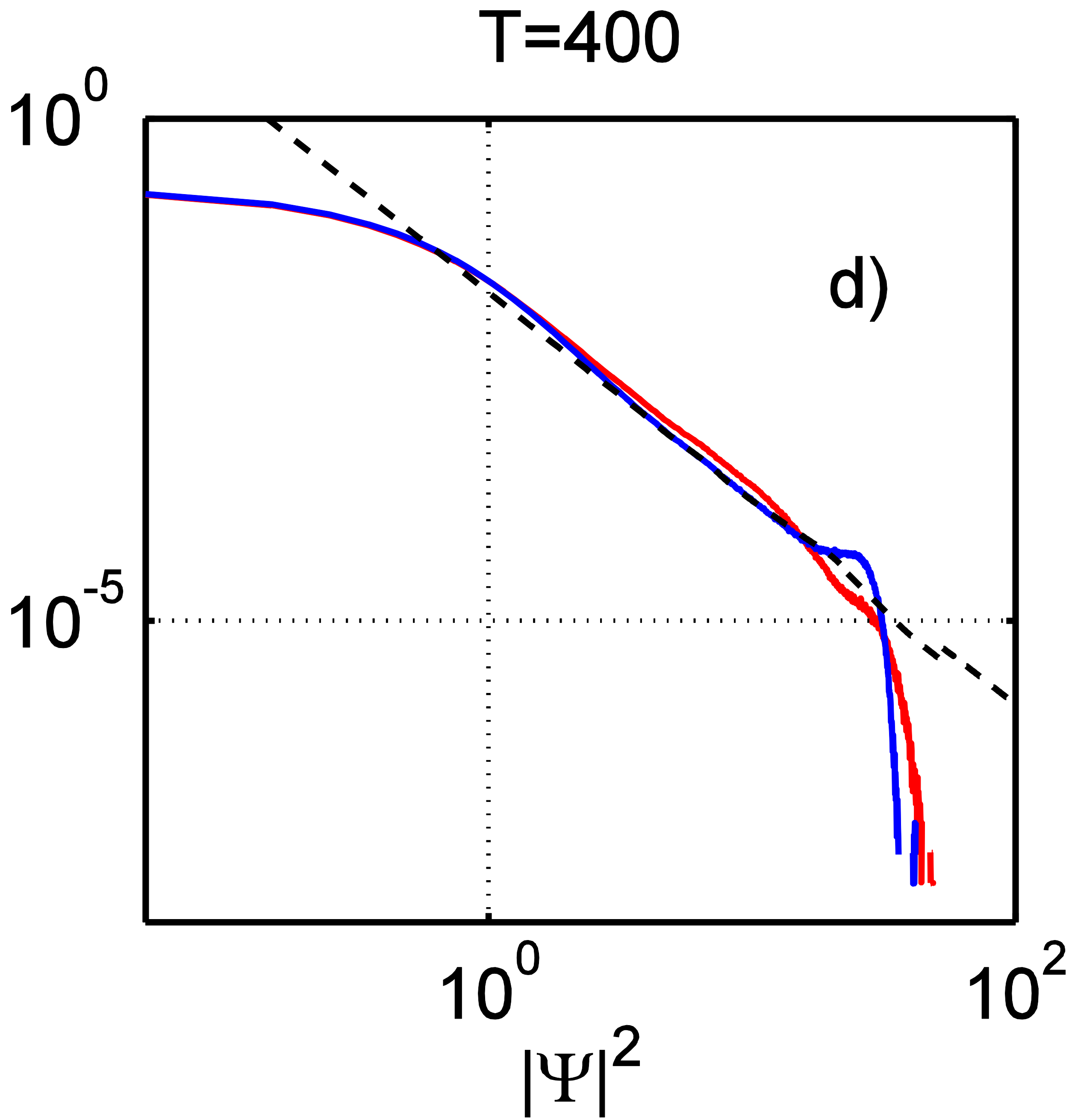

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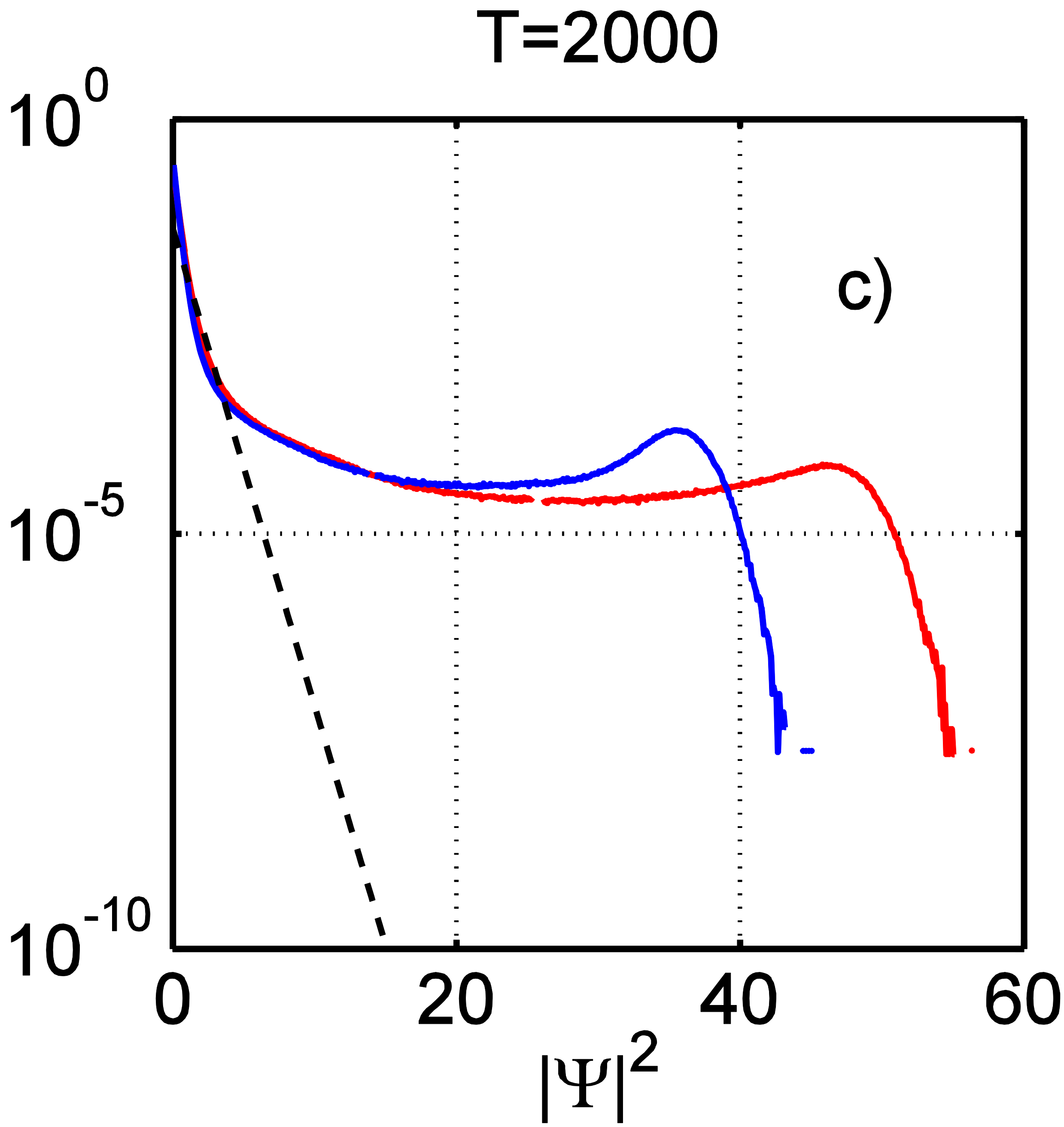

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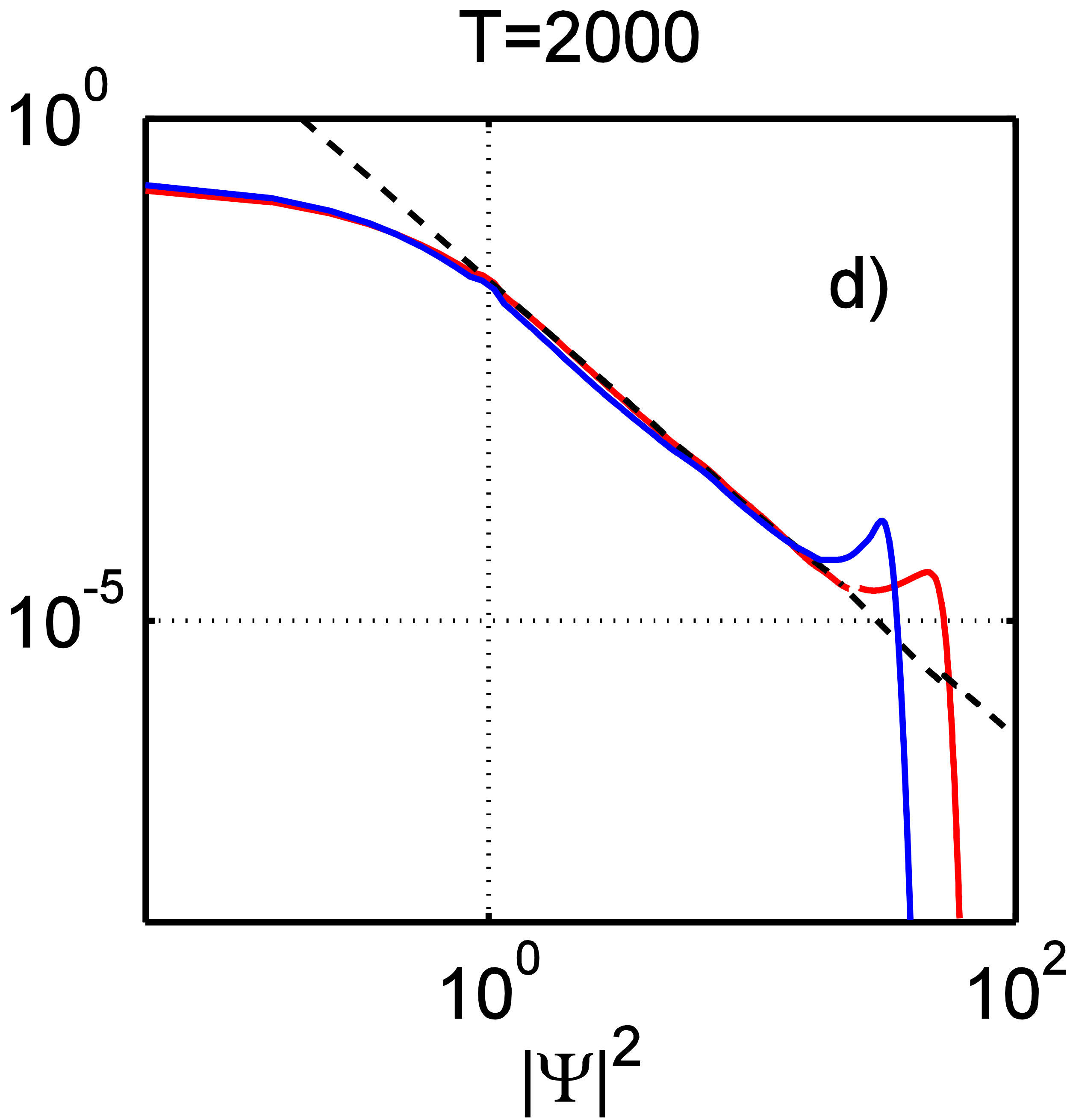
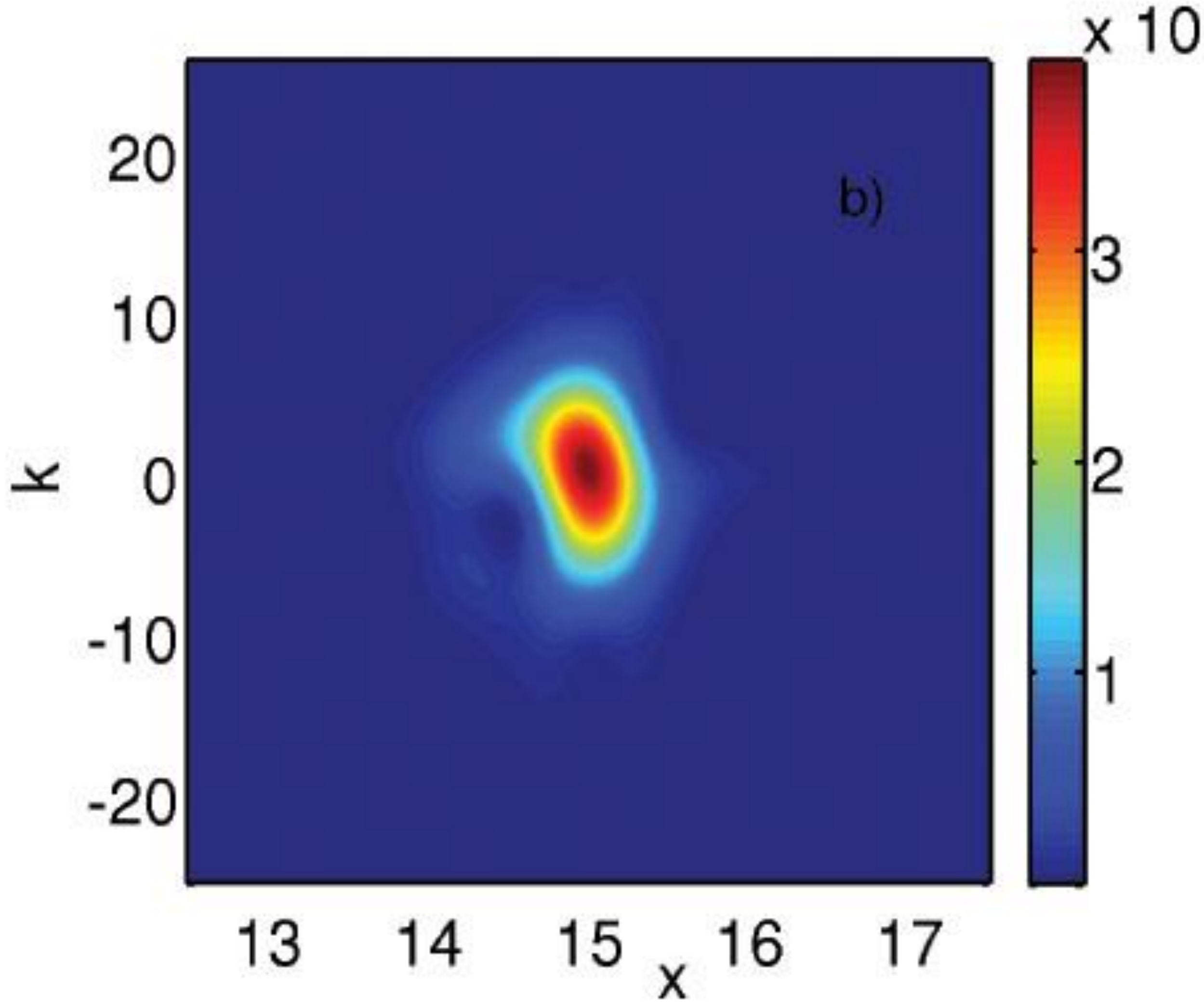
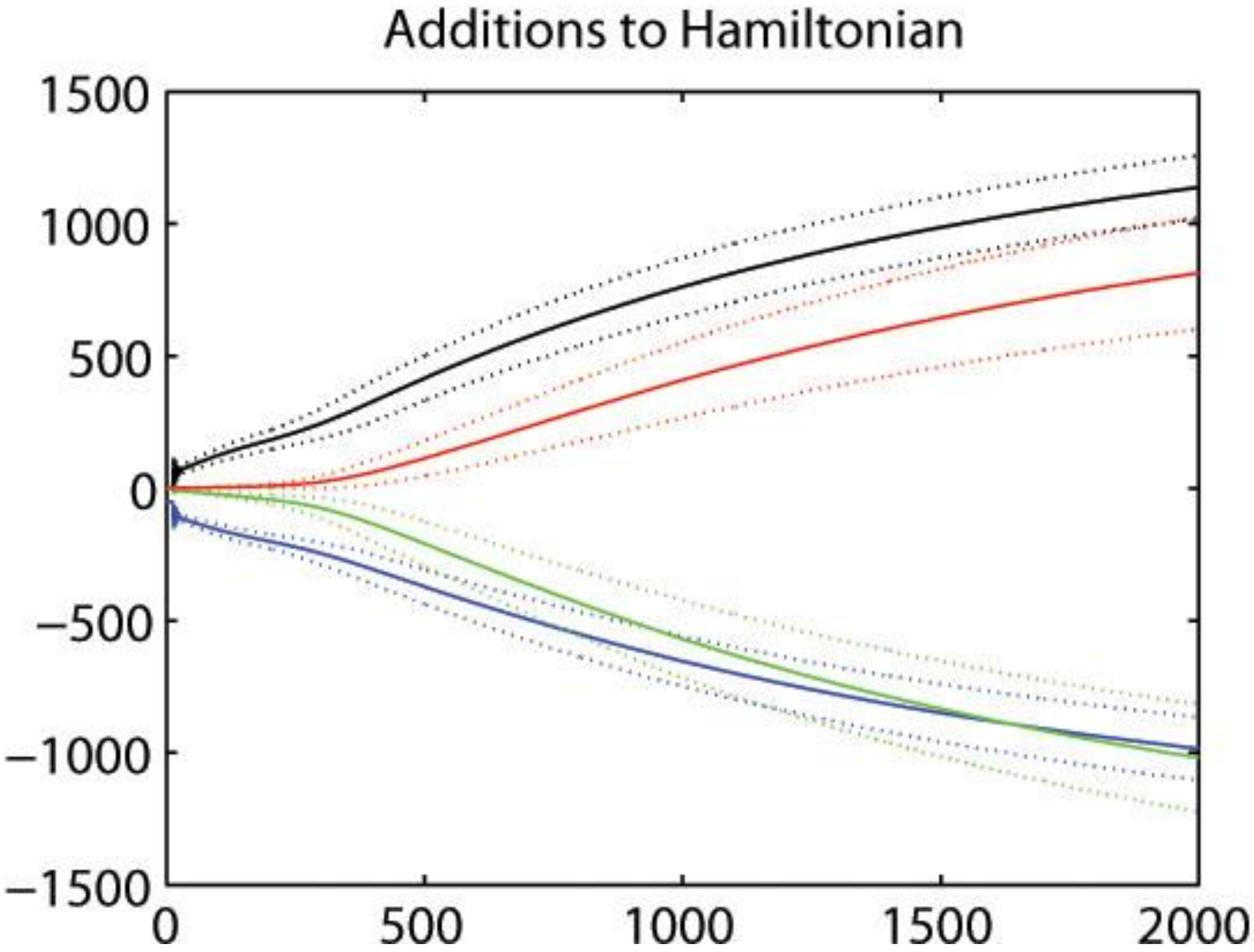


3.2. Generalized NLS equation accounting for 6- and 8-wave interactions: high waves are singular quasi-solitons.

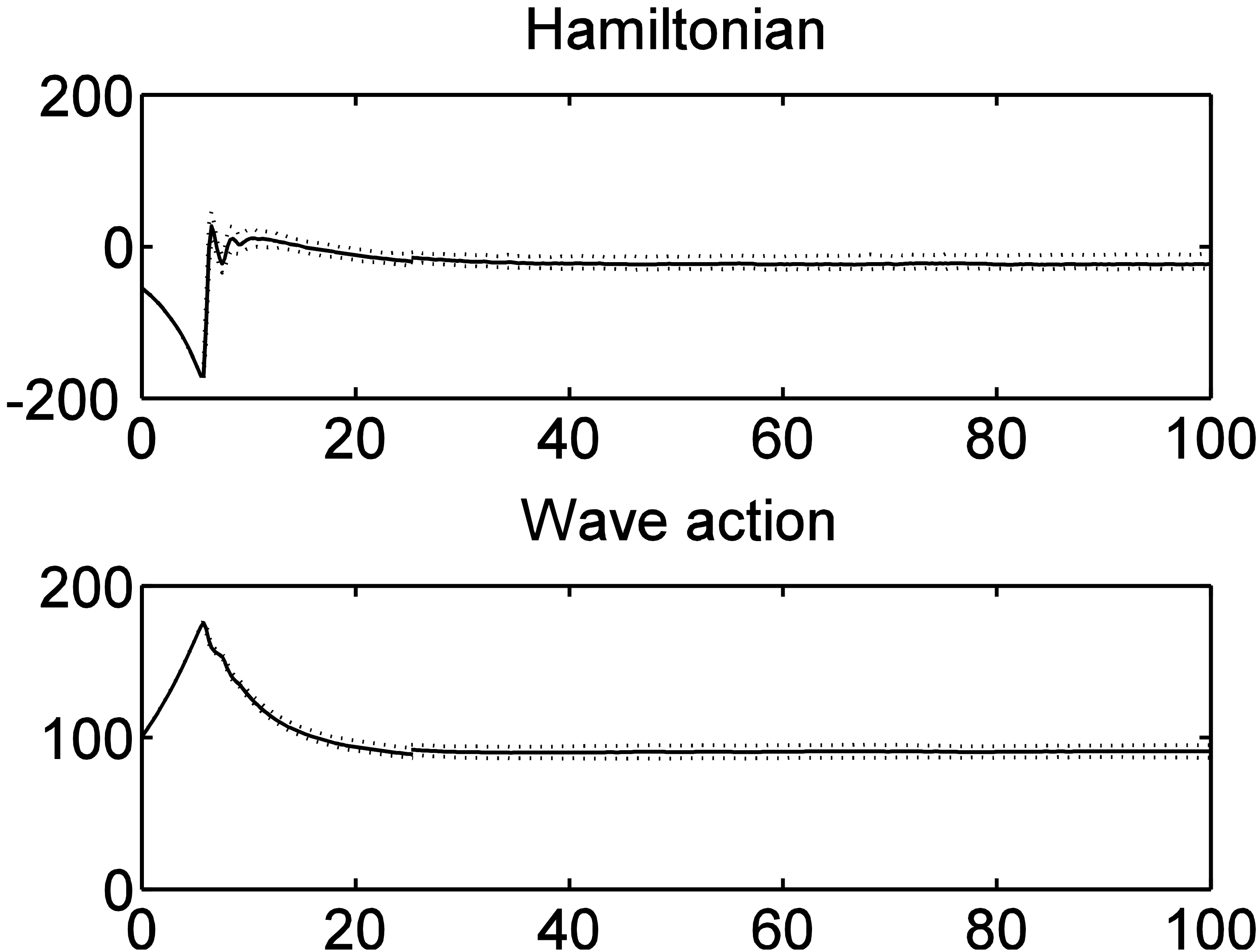


3.2. Generalized NLS equation accounting for 6- and 8-wave interactions: dynamics of parts of Hamiltonian.

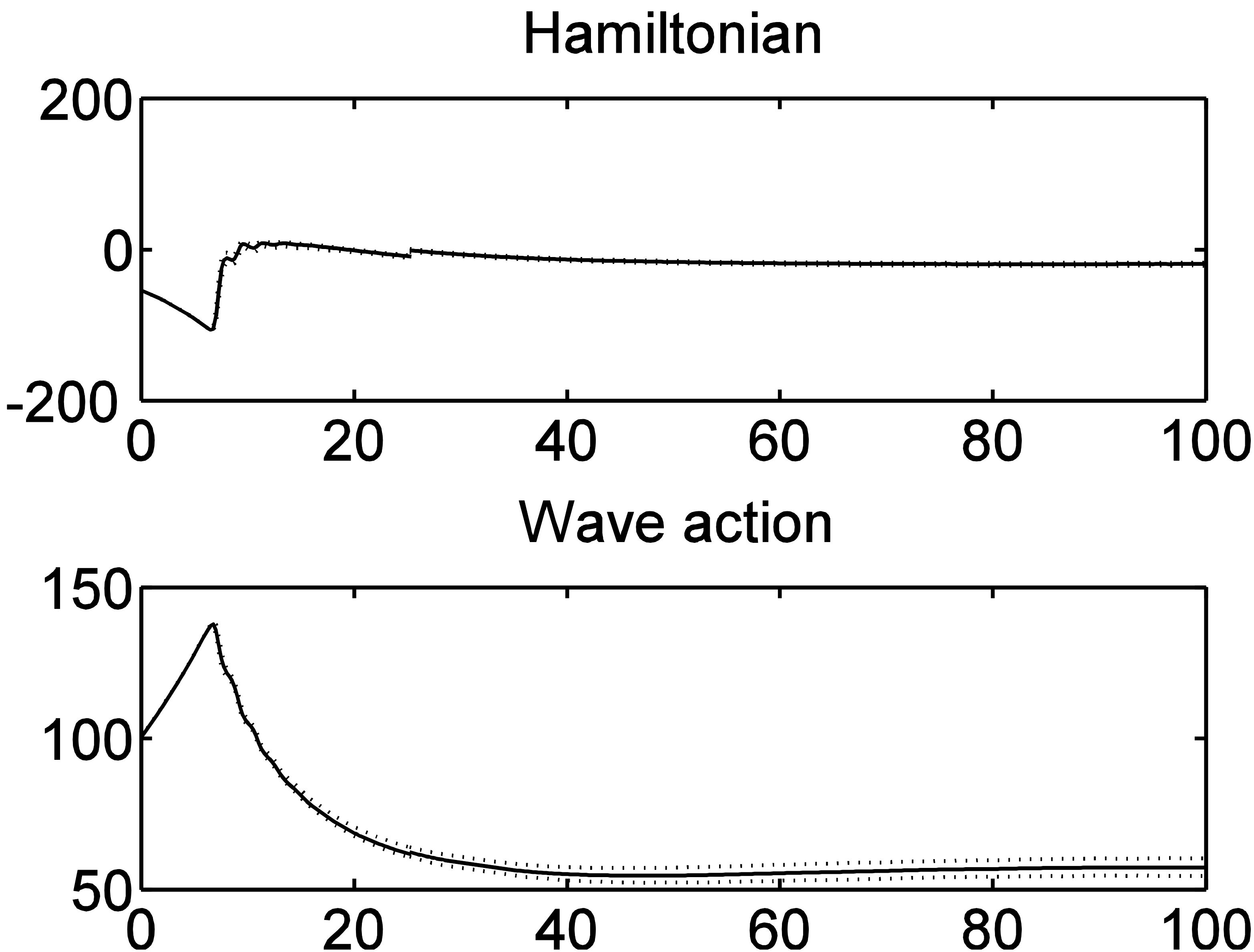


Evolution of H_d (black), H_{n1} (blue), H_{n2} (green), H_{n3} (red). Solid lines – median, dashed – standard deviations.

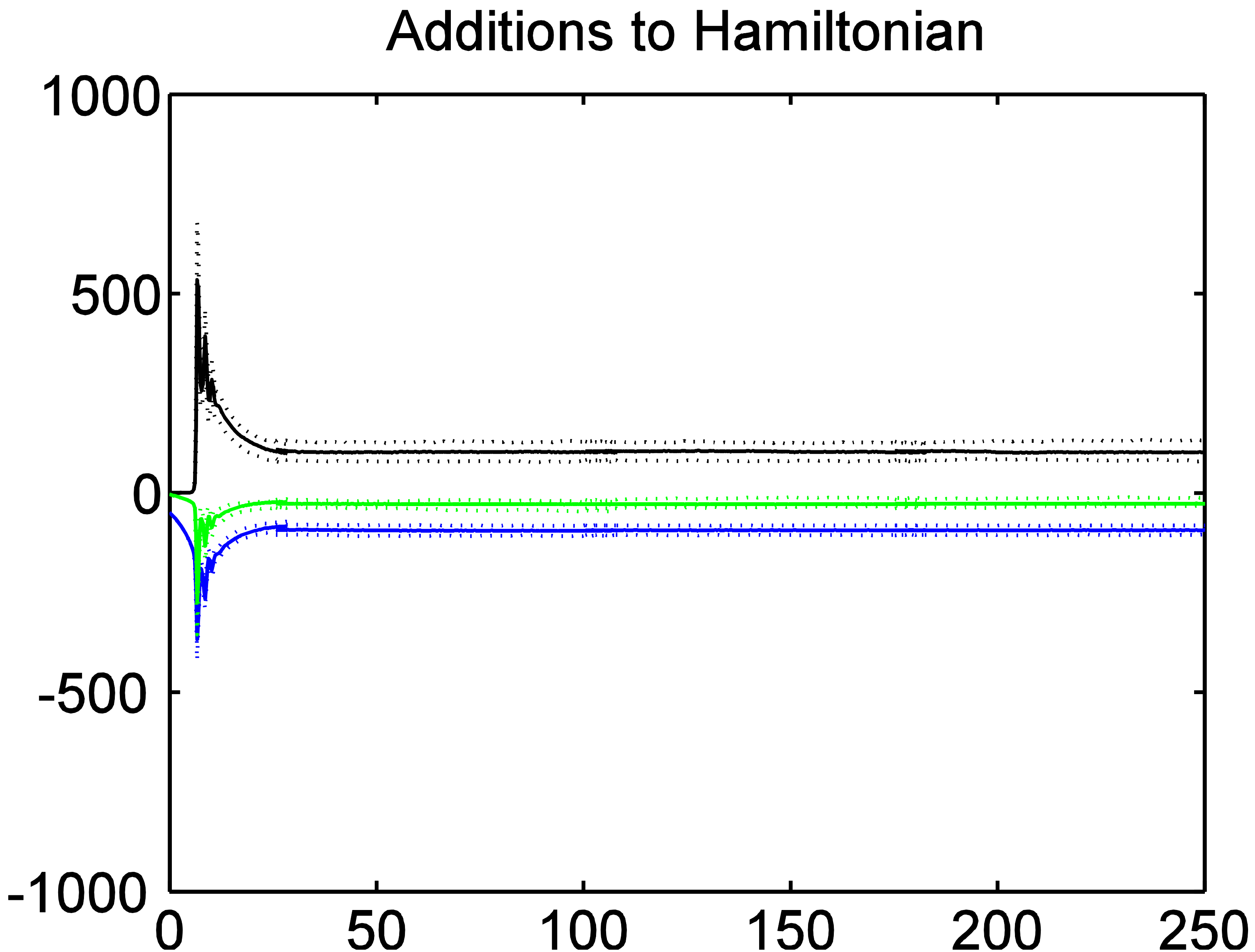
3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: dynamics of mean wave action and Hamiltonian for parameters **a=0.04, b=0.002, c=0.05, α=0.128**.



3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: dynamics of mean wave action and Hamiltonian for parameters **a=0.04, b=0.004, c=0.025, α=0.128**.

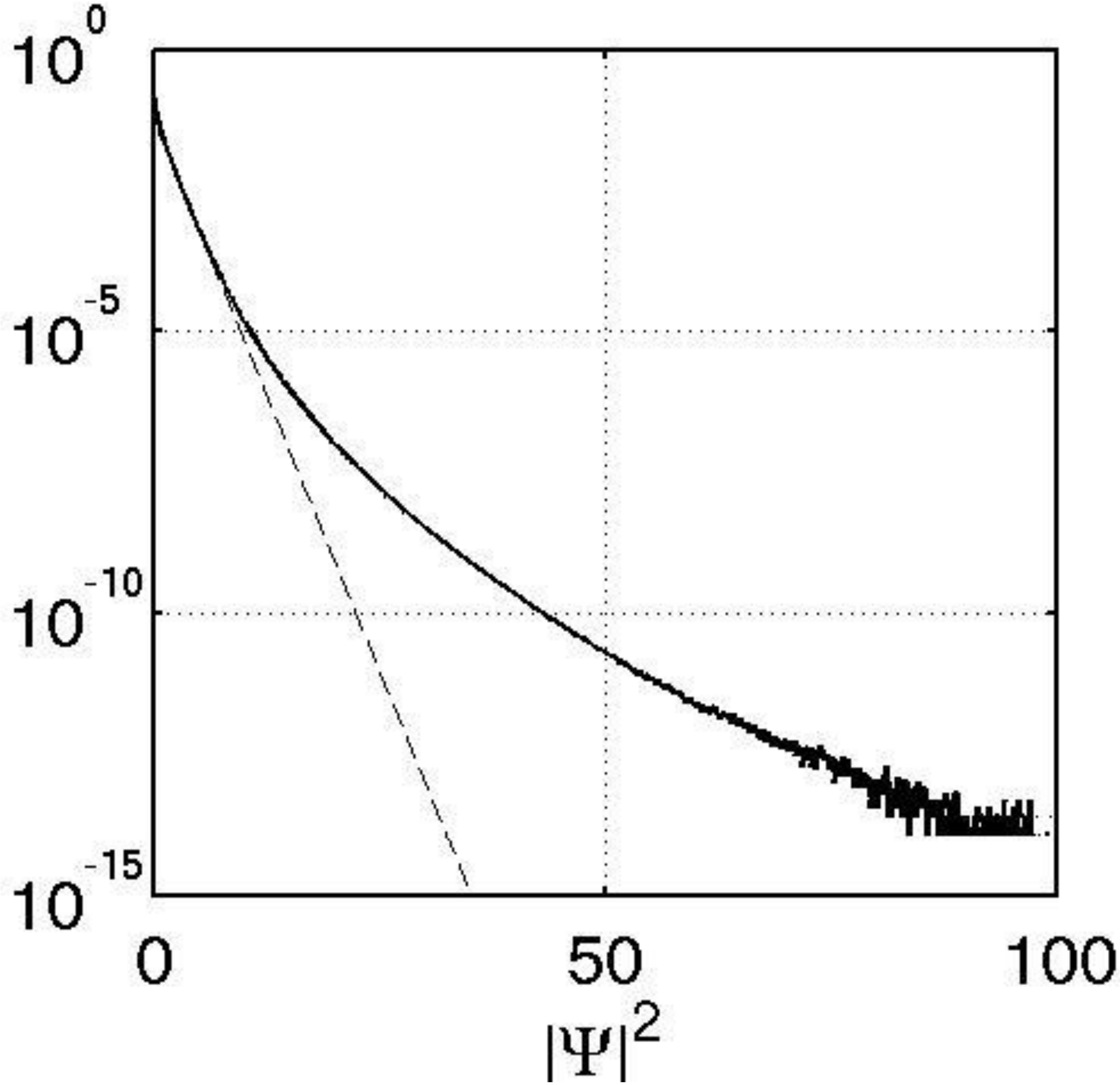


3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: dynamics of mean parts of Hamiltonian.

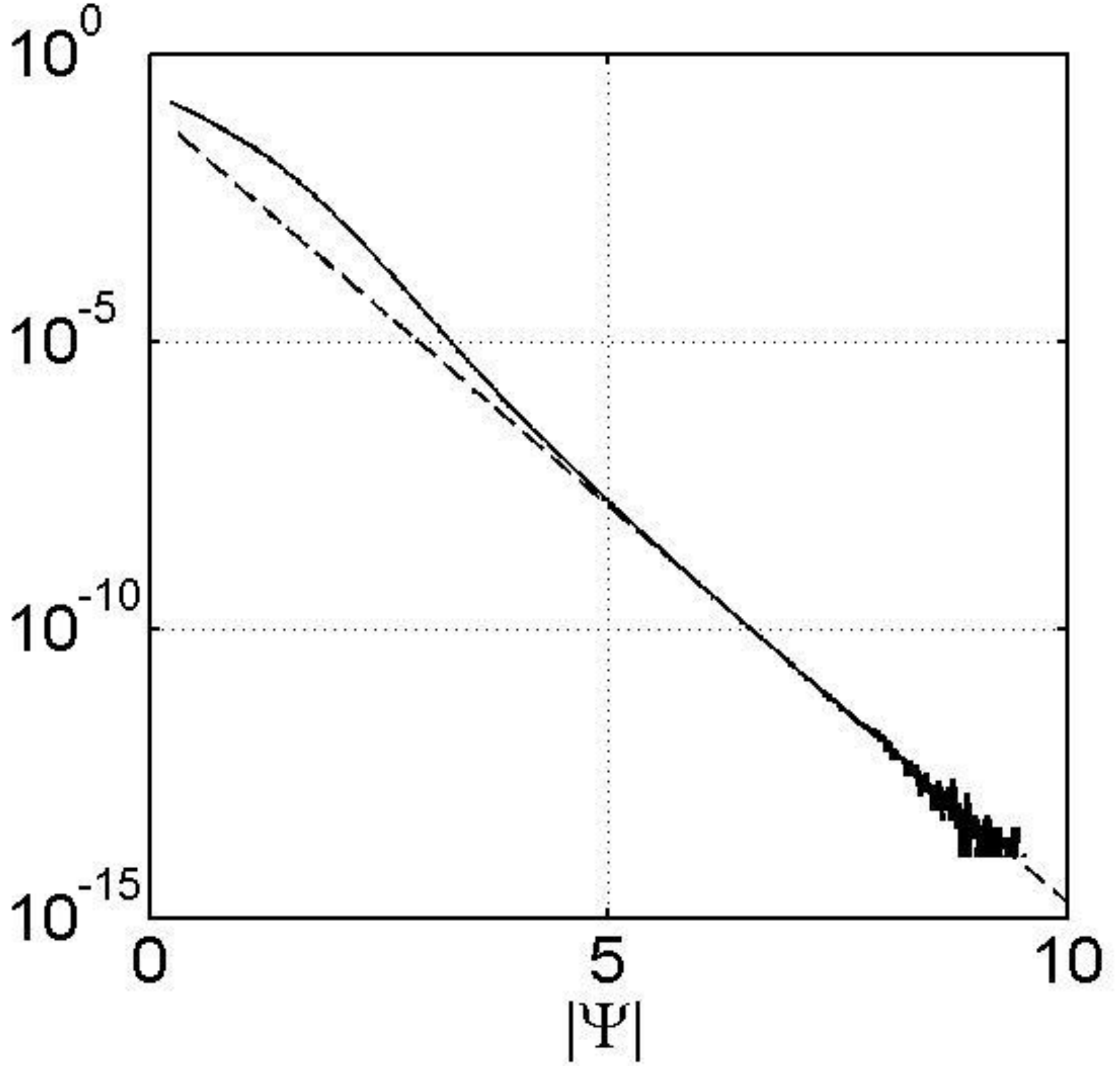


Evolution of H_d (black), H_{n1} (blue), H_{n2} (green). **a=0.04, b=0.004, c=0.05, α=0.128.**

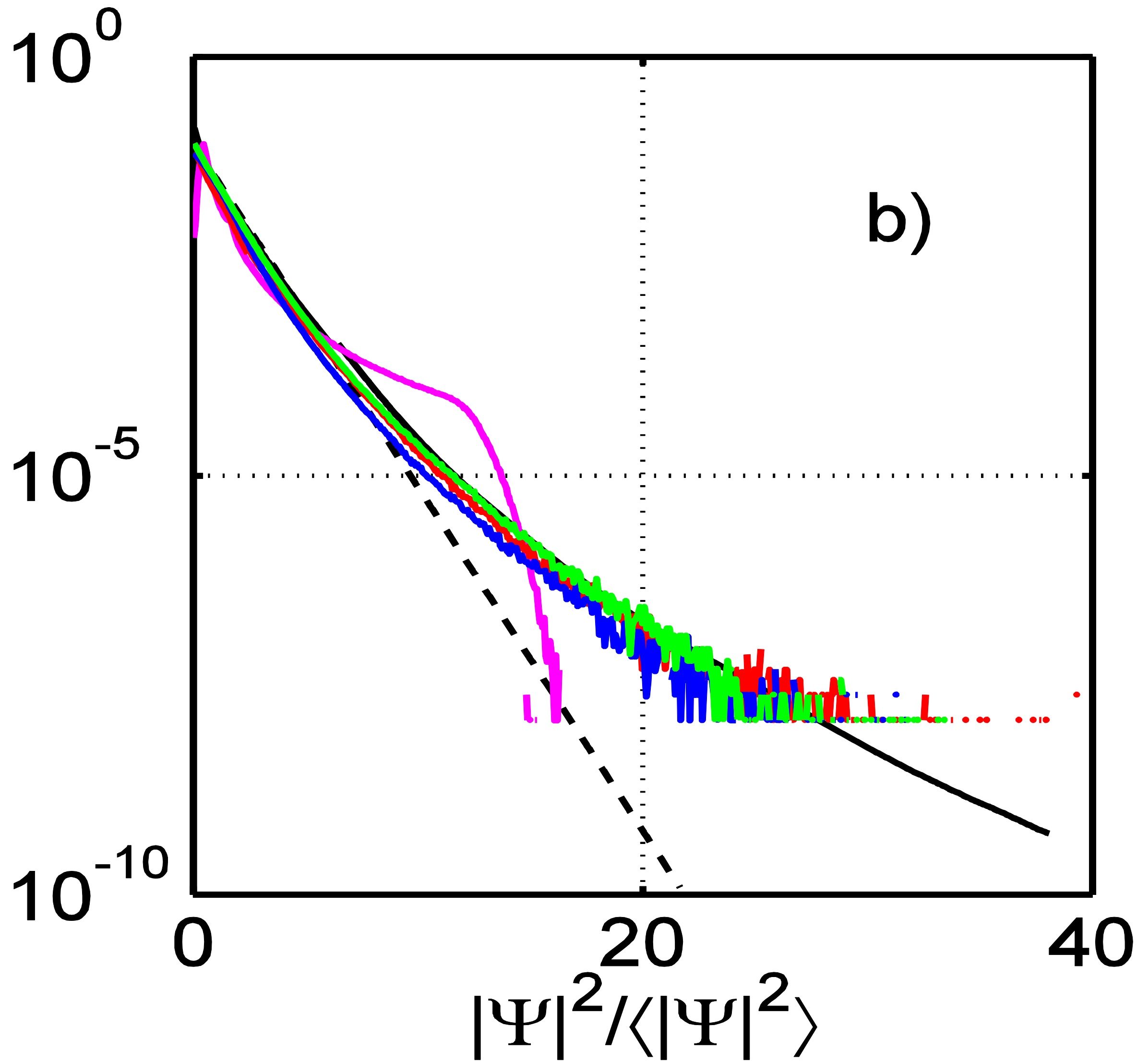
3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: deviations from Gaussian distribution, **a=0.04, b=0.004, c=0.05, α=0.128**.



3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: deviations from Gaussian distribution, **a=0.04, b=0.004, c=0.05, α=0.128**.

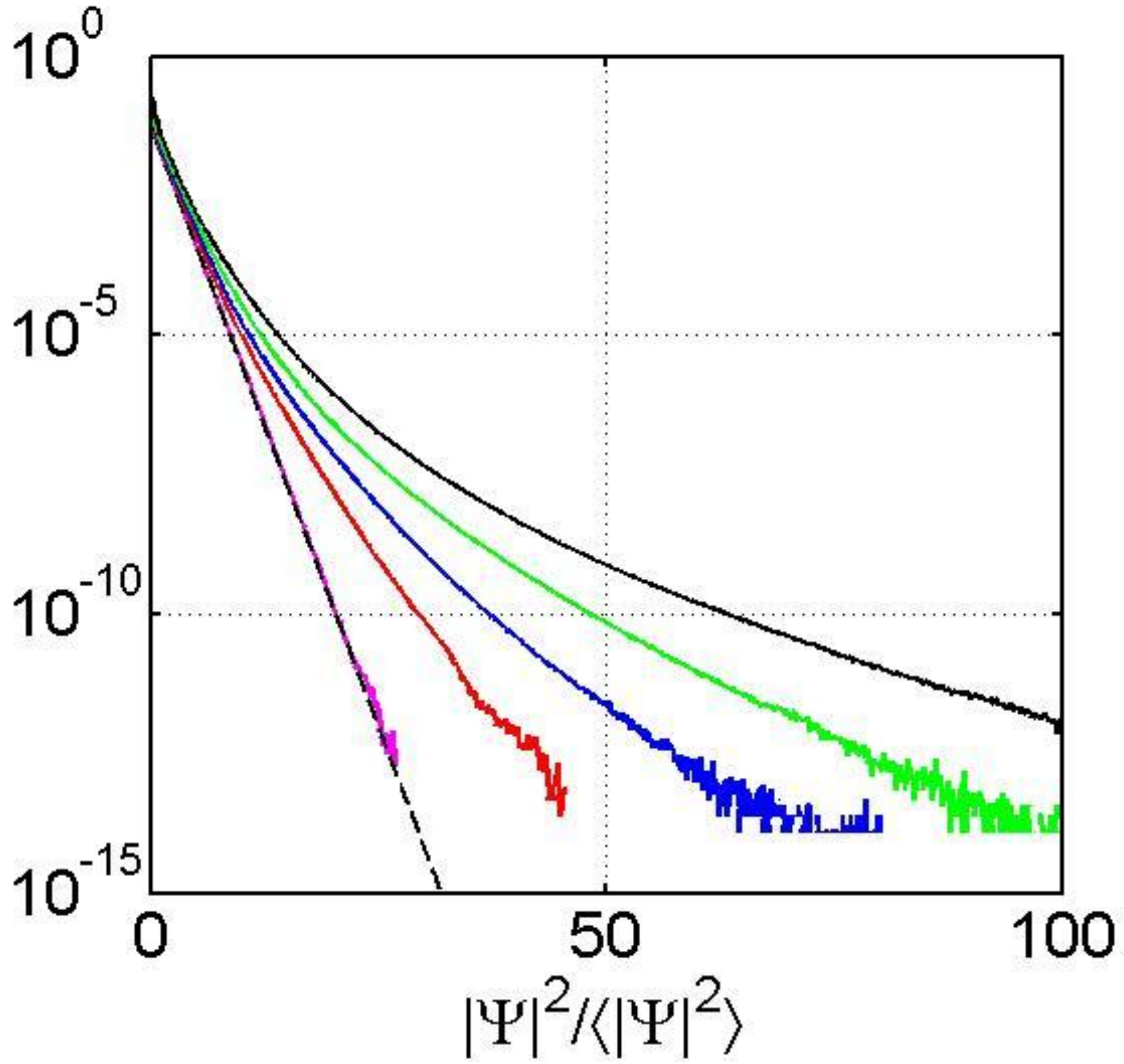


3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: deviations from Gaussian distribution, **a=0.04, b=0.004, c=0.05, α=0.128**.



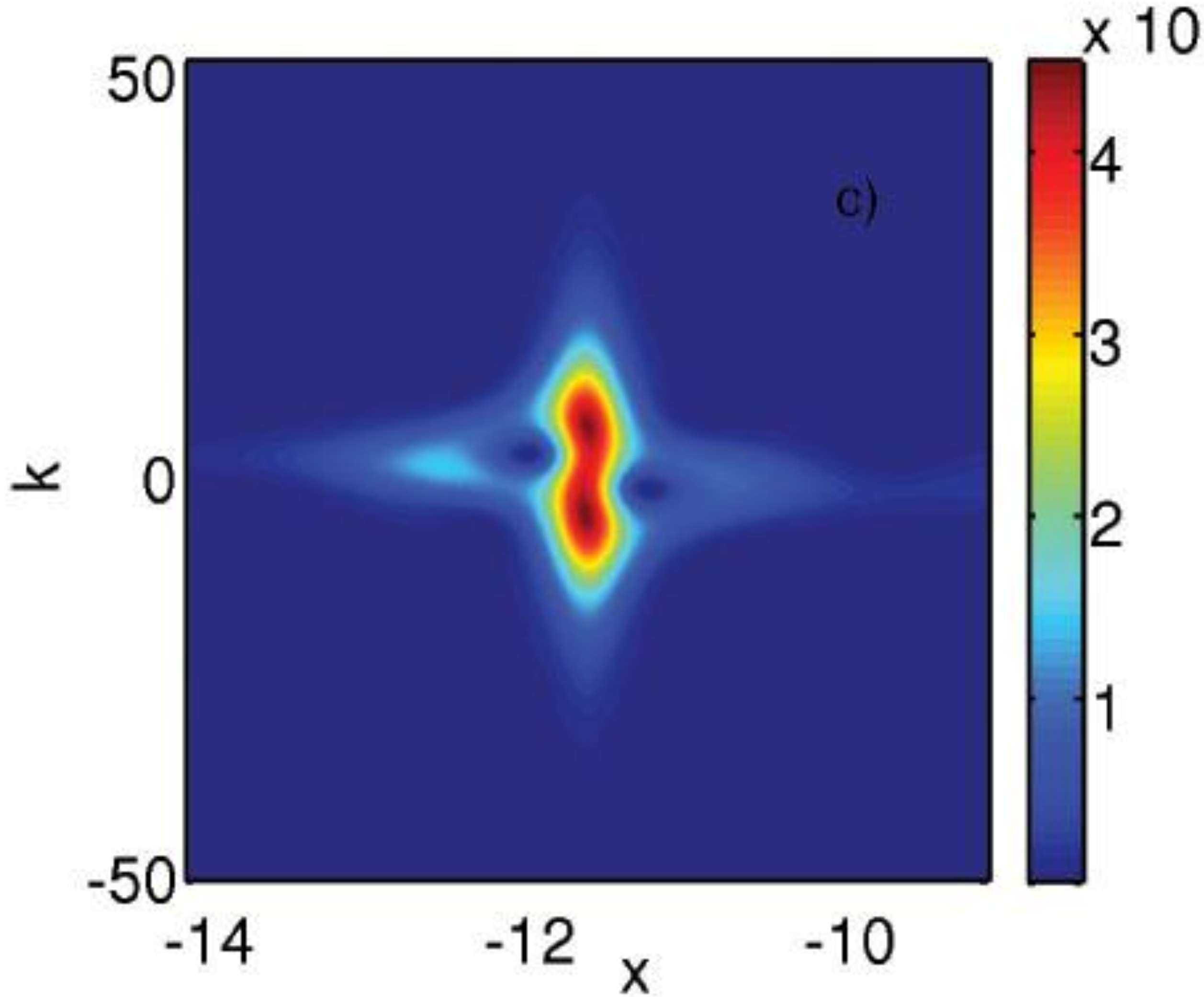
Statistics at time-shifts t=6 (purple), t=8 (red), t=10 (blue), t=12 (green), and in the statistically steady state (averaged from t=50 to t=250).

3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: deviations from Gaussian distribution, **a=0.04, b=0.004, c=0.05.**

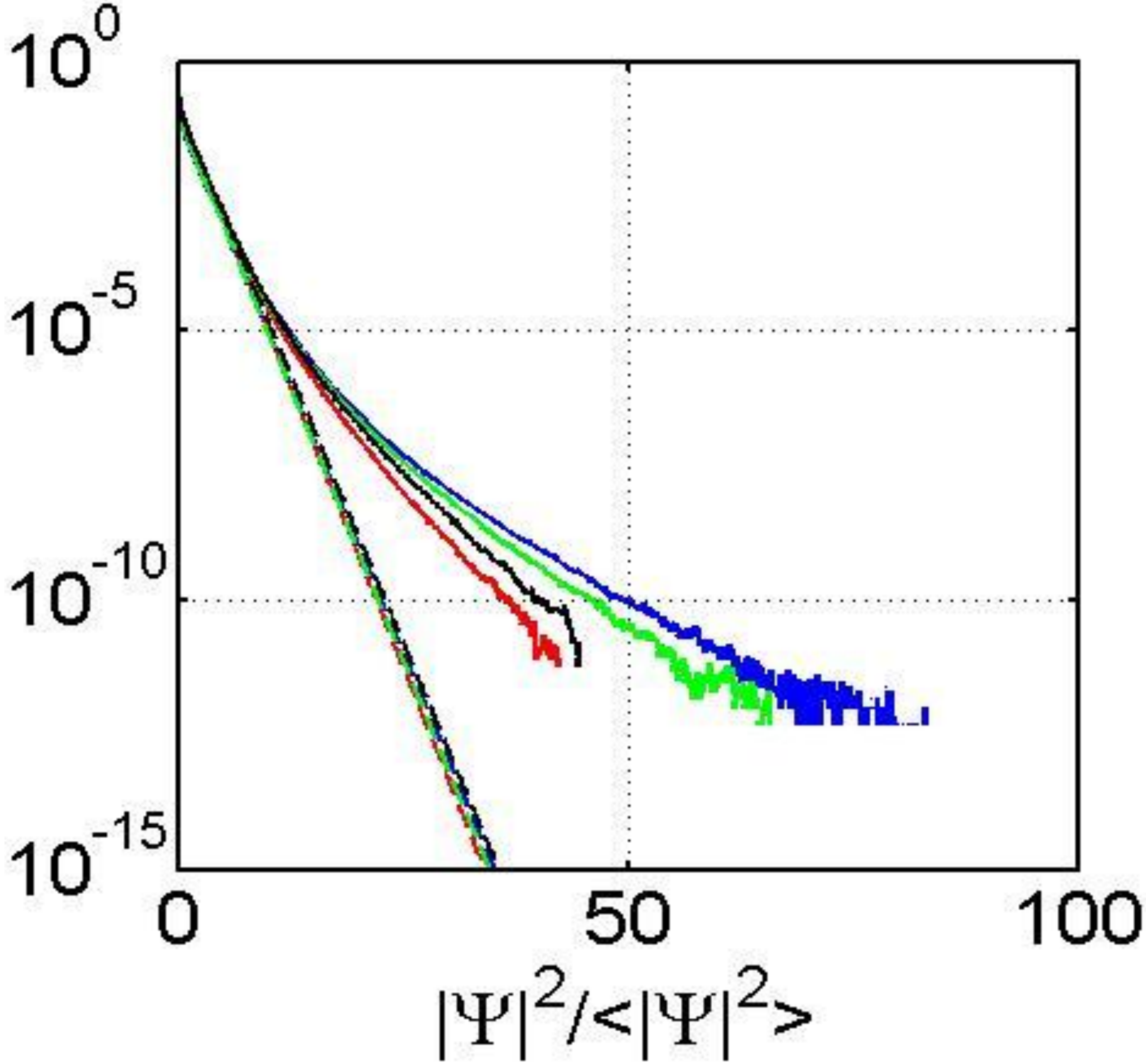


α=0 (purple), **α=0.032** (red), **α=0.064** (blue), **α=0.128** (green), **α=0.256** (black).

3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: high waves are collisions of quasi-solitons.



3.3. Generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms: deviations from Gaussian distribution, **α=0.128**.



Red line - **a=0.0784, b=0.004, c=0.05**. Blue line - **a=0.04, b=0.002, c=0.05**. Green line - **a=0.04, b=0.008, c=0.05**. Black line - **a=0.04, b=0.004, c=0.025**.

4. Results.

1. Gaussian statistics of waves does not necessarily mean linear system: as we demonstrated, classical NLS equation, and generalized NLS equation accounting for 6- and 8-wave interactions terms when 6- and 8-wave interactions are small both in significantly nonlinear regime have Gaussian decay of the corresponding PDFs.

2. We discovered existence of the power-law tails on the PDFs for the generalized NLS equation accounting for 6- and 8-wave interactions when 6- and 8-wave interactions are of the same order as for the terms of the classical NLS equation.

3. We discovered existence of non-Gaussian (exponential) tails on the PDFs for the generalized NLS equation accounting for 6-wave interactions, pumping and dumping terms. Non-Gaussian addition completely vanishes with absence of 6-wave interactions, and increase with 6-wave interactions coefficient.

Thank you for your attention!